Error Exponents for Backhaul-Constrained Parallel Relay Networks

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***Abstract*—In this paper, we assess the random coding error exponents (EEs) corresponding to decode-and-forward (DF), compress-and-forward (CF) and quantize-and-forward (QF) relaying strategies for a parallel relay network (PRN), consisting of a single source and two relays. Moreover, through numerical analysis we show that the EEs achieved by using QF relaying along with non-Gaussian signaling (coded modulation, M-QAM) at the source and symbol-by-symbol uniform scalar quantizers (uSQs) at the relays is better than that achieved by DF and CF relaying strategies when the system is in the low signal-to-noise ratio (SNR) regime and the backhaul capacity is sufficient. This behavior is due to the structure of coded modulation, as opposed to Gaussian signaling, which leads to better EEs for simple relaying strategies compared to its more complex counterparts.**

I. Introduction

For future mobile wireless networks one of the major concerns for service providers is to provide seamless connectivity to the end users with quality of service (QoS) as high as possible. One of the major hindrances to achieve a determined QoS is the interference caused by surrounding transmitters. In order to alleviate the interference effect in future cellular networks, base station cooperation (network MIMO) and relay deployment techniques have been recently proposed [1]–[5].

In this paper, we focus on a parallel relay network (PRN) consisting of a single source and two relays wherein an error-free finite capacity backhaul connection between the relays and destination is assumed. This model was first studied by Schein [6] where he derived several outer bounds and achievable rates. This setup can find *applications* in cellular networks for UL communications, in long-range sensor networks, and in rapidly deployable infrastructure networks for military or civil applications. The impact of limited-capacity backhaul on both base station and mobile station cooperation for uplink and downlink for non-fading Gaussian scenarios have been studied in [4], [5].

Note that system performance is highly dependent on the processing capabilities of RSs. In this paper, we investigate whether it is possible to have good performance by using simple and cheap relays with limited backhaul connections to the destination. In particular, we look at a *simpler* and more *practical* quantization technique at the relays which relies on symbol-by-symbol uniform scalar quantization (uSQ), since in the high resolution regime the performance loss compared to vector quantization (VQ) becomes negligible [9].

In order to have thorough characterization of a system's performance, knowing the capacity of the system is not sufficient alone. Hence, in this paper, we consider the random coding error exponent [8], which is also defined as channel reliability function and represents a decaying rate in the decoding error probability as a function of codeword length, as the performance metric. In particular, we assess the random coding EEs corresponding to DF, CF and QF relaying strategies for the PRN setup. Specifically, for the DF we assume Gaussian codebook at the source and maximum-likelihood (ML) decoding at the relays where each passes its own decision and a corresponding *reliability function* to the destination. For the CF, we assume Gaussian codebook at the source and VQ at the relays and ML decoding at the destination. For the QF, we assume M-QAM at the source and uSQ at the relays. Moreover, through numerical analysis we show that the EEs achieved by using the proposed QF relaying along with M-QAM at the source and simple symbol-by-symbol uSQ at the relays is better than that of DF and CF relaying strategies when the system is in the low SNR regime and the backhaul capacity is large enough.

II. Channel Model and Preliminaries

We study the PRN model shown in Fig. 1 where a single source wants to communicate with a destination with the assistance of two relay stations (RSs). We assume no link between the source and destination nor between the RSs. All the channels are modeled as time-invariant, memoryless additive white Gaussian noise (AWGN) channels with constant gain (which may correspond to path-loss between each transmitter and receiver). The source encodes its message $W \in \mathcal{W}$, where $\mathcal{W} = \{1, 2, \ldots, 2^{nR}\}$ and R is the transmission rate in [bits/transmission], into the codeword $X^n(W)$.

The received signal at the k-th RS after the i-th channel use, for $k = 1, 2$ and $i \in [1, n]$, is [1]

$$Y_{R_k,i} = h_k X_i + Z_{k,i}, \tag{1}$$

where $h_k \in \mathbb{R}^+$ is the fixed channel gain from the source to the k-th RS, $Z_{k,i} \sim \mathcal{CN}(0, \sigma^2)$ is the noise term at the k-th

[1]We use capital letters, e.g., X, for random variables (RVs), lower case letters, e.g., x, for the realization of these RVs, and calligraphic letters, e.g., $\mathcal{X}$, for their alphabets. Also, $\mathbb{E}[(.)]$ denotes the expectation operator, $\mathbf{I}_m$ is the $m \times m$ identity matrix. $X \sim \mathcal{CN}(\mu, \sigma^2)$ means RV X follows circularly symmetric complex Gaussian distribution with mean μ and variance σ^2.

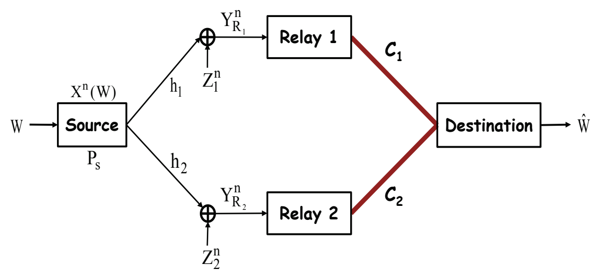

Fig. 1. A single source, 2 relay PRN setup with orthogonal error-free finite-capacity backhaul links between the relays and the destination, where C_k in [bits/transmission] is the link capacity between the k-th relay and the destination, for $k = 1, 2$.

relay. We assume an average power constraint at the source, i.e., $\mathbb{E}[|X(W)|^2] = P_s$, $\forall W \in \mathcal{W}$.

The k-th RS transmits X_{R_k} based on the previously received signals (causal encoding) [7]

$$X_{R_k,i} = f_{R_k,i}(Y_{R_k,1}, Y_{R_k,2}, \dots, Y_{R_k,i-1}) \tag{2}$$

where $i \in [1, n]$ is the time index.

For the access channel from the RSs to the destination, we consider lossless orthogonal links with finite capacity between each RS and the destination. Let C_k[bits/transmission], $k = 1, 2$, be the link capacity between the k-th RS and the destination.

A. Random Coding Error Exponent

In order to have thorough characterization of a system's performance, the capacity of the system alone is not sufficient. The random coding EE [8], which is also defined as channel reliability function and represents a decaying rate in the decoding error probability as a function of codeword length, gives insights about how to achieve a certain level of reliability in communication at a rate below the channel capacity.

The error exponent of a communication system is defined by [8]

$$E(R) \triangleq \lim_{n\to\infty} \sup \frac{-\log_2 P_e(n, R)}{n} \tag{3}$$

where $P_e(R, n)$ is the average block error probability for the optimum block code of length n and rate R[bits/transmission]. For any rate below capacity, the average probability of decoding error $P_e(R, n)$ for codes of block length n can be bounded between the limits

$$2^{-n[E_{sp}(R)+O(n)]} \leq P_e(n, R) \leq 2^{-nE_r(R)} \tag{4}$$

where $E_{sp}(R)$, known as *sphere packing* EE, and $E_r(R)$, known as *random coding* EE, are lower and upper bounds on the reliability function $E(R)$, respectively, and $O(n)$ is a function going to 0 with increasing n. For a given code $\mathcal{C}$ of length n and alphabet size 2^{nR}, Gallager's random coding EE, which relies on ML decoding, is given by

$$E_r(R) = \max_{0\leq\rho\leq1} \max_{\mathbf{p}} [E_0(\rho, \mathbf{p}) - \rho R] \tag{5}$$

where $E_0(\rho, \mathbf{p})$ is defined as

$$E_0(\rho, \mathbf{p}) = -\log_2 \left[\sum_{\mathbf{y}} \left(\sum_{\mathbf{x}} p(\mathbf{x}) p(\mathbf{y}|\mathbf{x})^{\frac{1}{1+\rho}} \right)^{1+\rho} \right] \tag{6}$$

for discrete channels where $p(\mathbf{x})$ is the input distribution and $p(\mathbf{y}|\mathbf{x})$ are the channel output distributions conditioned on the input, and

$$E_0(\rho, \mathbf{f}) = -\log_2 \left[\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(\mathbf{x}) f(\mathbf{y}|\mathbf{x})^{\frac{1}{1+\rho}} d\mathbf{x} \right)^{1+\rho} d\mathbf{y} \right] \tag{7}$$

for AWGN channels where $f(\mathbf{x})$ is the continuous input distribution and $f(\mathbf{y}|\mathbf{x})$ the channel output distributions conditioned on the input.

III. Error Exponent Analysis for Single User PRN

In this section, we obtain expressions for the EEs corresponding to DF, CF and QF relaying strategies for the considered PRN set-up. For the DF, we assume a Gaussian codebook at the source and ML decoding at the RSs where each passes its own decision and a corresponding *reliability function* to the destination. We note that for the DF the destination is not required to have channel side information (CSI). For the CF, we use a Gaussian codebook at the source and VQ at the RSs and ML decoding at the destination. For the proposed QF relaying, M-QAM at the source and uSQ at the RSs are considered.

A. DF relaying with Gaussian Inputs

Assume each RS applies ML detection and sends the message corresponding to the detected signal along with a *reliability information* (which is a scalar variable equal to the logarithm of the Euclidean distance between the received signal and the detected signal) to the destination on orthogonal error- and cost-free limited capacity backhaul links. Moreover, we assume that the backhaul link capacities are at least equal to the source transmission rate, R. Hence, the backhaul links do not create a bottleneck for system performance.

Upon receiving the detected signals and the reliability information, the destination makes its decision by comparing the reliability information: it decides on the codeword which has the minimum reliability information (Euclidean distance). Hence, if the codeword detected at one of the RS is wrong and its corresponding reliability information is smaller, then the ultimate detection will be wrong even if the other RS has made a correct detection (but with greater reliability information).

Assume the w-th message, $w \in \mathcal{W}$, is encoded into the codeword $\mathbf{x}(w) \in \mathbb{C}^n$ of length n and let $\mathbf{y}_{R_k} \in \mathbb{C}^n$ denote the received signal vector of size n at the k-th RS for $k = 1, 2$. Then, the ML detection at the k-th RS is given by

$$\begin{aligned} \hat{\mathbf{x}}_{ML,k} &= \arg\max_{\mathbf{x}} \ln \left(p(\mathbf{y}_{R_k}|\mathbf{x}(w), h_k) \right) \\ &= \arg\max_{\mathbf{x}} -\frac{1}{\sigma^2} \|\mathbf{y}_{R_k} - h_k \mathbf{x}\|^2 - n \ln(\pi\sigma^2) \\ &= \arg\min_{\mathbf{x}} \|\mathbf{y}_{R_k} - h_k \mathbf{x}\| = \arg\min_{\mathbf{x}} \beta_k, \qquad k = 1, 2, \end{aligned}$$

where we define β_k as *the reliability information,* i.e.,

$$\beta_k = \|\mathbf{y}_{R_k} - h_k \mathbf{x}\|, \qquad k = 1, 2.$$

Upon receiving the detected signal and the reliability information of each RS, the destination node makes the following final detection:

$$\hat{\mathbf{x}}_{ML} = \arg \min_{\hat{\mathbf{x}}_{ML,1}, \hat{\mathbf{x}}_{ML,2}} \beta_k, \qquad k = 1, 2. \tag{8}$$

We define $\mathcal{E}_a \triangleq \Pr\{\beta_a > \beta_b | \hat{\mathbf{x}}_{ML,b} \neq \mathbf{x}(w), \hat{\mathbf{x}}_{ML,a} = \mathbf{x}(w)\}$ for $a, b = 1, 2, a \neq b$. Then, with the above detection rule we have the following average probability of error (conditioned on $\mathbf{x}(w)$ was sent)

$$\begin{aligned} P_e &\leq P_{ML,1}P_{ML,2} + P_{ML,1}(1 - P_{ML,2}) \Pr\{\mathcal{E}_2\} \\ &\qquad\qquad + P_{ML,2}(1 - P_{ML,1}) \Pr\{\mathcal{E}_1\} \\ &\leq P_{ML,1}P_{ML,2} + P_{ML,1} \Pr\{\mathcal{E}_2\} + P_{ML,2} \Pr\{\mathcal{E}_1\} \end{aligned} \tag{9}$$

where $P_{ML,k}$, for $k = 1, 2$, is the standard ML error probabilities at the k-th relay.

Assuming *symmetric* channels from the source to the RSs, i.e., $h = h_1 = h_2$, and hence $P_{ML} = P_{ML,k}$ and $\Pr\{\mathcal{E}_1\} = \Pr\{\mathcal{E}_2\}$, the probability of error will have the following simplified expression:

$$P_e \leq P_{ML}^2 + 2P_{ML} \Pr\{\mathcal{E}_2\}. \tag{10}$$

Now we need to find an expression for $\Pr\{\mathcal{E}_2\}$. This can be evaluated as follows

$$\begin{aligned} \Pr\{\mathcal{E}_2\} &= \Pr\{\beta_2 > \beta_1 \mid \hat{\mathbf{x}}_{ML,1} \neq \mathbf{x}(w), \hat{\mathbf{x}}_{ML,2} = \mathbf{x}(w)\} \\ &= \Pr\{\|\mathbf{y}_{R_1} - h_1\hat{\mathbf{x}}_{ML,1}\|^2 \leq \|\mathbf{y}_{R_2} - h_2\hat{\mathbf{x}}_{ML,2}\|^2\} \\ &= \Pr\{\|\mathbf{y}_{R_1} - h_1\hat{\mathbf{x}}_{ML,1}\|^2 \leq \|\mathbf{y}_{R_2} - h_2\mathbf{x}\|^2\} \\ &= \Pr\{\|h_1(\mathbf{x} - \hat{\mathbf{x}}_{ML,1}) + \tilde{\mathbf{z}}_1\|^2 \leq \|\tilde{\mathbf{z}}_2\|^2\} \\ &= \Pr\{\|\hat{\mathbf{z}}_1\|^2 - \|\tilde{\mathbf{z}}_2\|^2 \leq 0\} \\ &= \Pr\{T - Y \leq 0\} = \Pr\{Z \leq 0\} \end{aligned} \tag{11}$$

where $\hat{\mathbf{z}}_1 \triangleq h_1(\mathbf{x} - \hat{\mathbf{x}}_{ML,1}) + \tilde{\mathbf{z}}_1 \sim \mathcal{CN}(\mathbf{0}, (2h_1^2 P_s + \sigma^2)\mathbf{I}_n)$, we note that Gaussian codebook is assumed at the source with P_s being the average source power, and $\tilde{\mathbf{z}}_2 \sim \mathcal{CN}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$. Furthermore, we define the random variables (RVs) $T \triangleq \|\hat{\mathbf{z}}_1\|^2$, $Y \triangleq \|\tilde{\mathbf{z}}_2\|^2$ and $Z \triangleq T - Y$.

The RV Z can be re-written in the following form

$$\begin{aligned} Z &= T - Y = \|\hat{\mathbf{z}}_1\|^2 - \|\tilde{\mathbf{z}}_2\|^2 \\ &= \sum_{i=1}^{n} \left(|\hat{z}_{1,i}|^2 - |\tilde{z}_{2,i}|^2\right) = \sum_{i=1}^{n} (T_i - Y_i) = \sum_{i=1}^{n} Z_i \end{aligned} \tag{12}$$

where[2] $T_i \triangleq |\hat{z}_{1,i}|^2 \sim \text{Exp}(\lambda_t)$ with $\lambda_t = 1/(2h_1^2 P_s + \sigma^2)$, $Y_i \triangleq |\tilde{z}_{2,i}|^2 \sim \text{Exp}(\lambda_y)$ with $\lambda_y = 1/\sigma^2$, and $Z_i = T_i - Y_i$, $i = 1, \ldots, n$. With these definitions the probability distribution function (pdf) of Z_i $f_{Z_i}(z_i)$ is given by (23) (see Appendix-A for the derivations of the pdf) with mean $\mu_Z = \mathbb{E}[Z_i]$ and variance $\sigma_Z^2 = \mathbb{VAR}[Z_i]$, for $i = 1, \ldots, n$.

[2]The notation $T \sim \text{Exp}(\lambda)$ means that T is an exponentially distributed RV with mean λ, i.e., $p_T(t) = \lambda \exp\{-\lambda t\},\ t \geq 0$.

Finally, we can upper bound $\Pr\{\beta_2 > \beta_1\}$ as follows[3], for sufficiently large n,

$$\begin{aligned} \Pr\{\beta_2 > \beta_1\} &= \Pr\{\beta_2 > \beta_1 | \hat{\mathbf{x}}_{ML,1} \neq \mathbf{x}(w), \hat{\mathbf{x}}_{ML,2} = \mathbf{x}(w)\} \\ &= \Pr\{Z \leq 0\} = \Pr\left\{\sum_{i=1}^{n} Z_i \leq 0\right\} \\ &\overset{(a)}{\leq} Q\left(\frac{n\mu_Z}{\sqrt{n\sigma_Z^2}}\right) \overset{(b)}{\leq} \exp\left\{-n\frac{\mu_Z^2}{2\sigma_Z^2}\right\} \\ &\leq \exp\left\{-n\left(\frac{1}{2} - \frac{\lambda_t^2\lambda_y^2}{\lambda_t^4 + \lambda_y^4}\right)\right\} \\ &\overset{(c)}{=} 2^{-n\left(\frac{\log_2(e)}{2} - \frac{\log_2(e)(1+2\Gamma)^2}{1+(1+2\Gamma)^4}\right)} \end{aligned} \tag{13}$$

where $\Gamma = \frac{h^2 P_s}{\sigma^2}$ with $h = h_1 = h_2$, (a) follows from the central limit theorem, (b) follows by upper-bounding the standard tail function $Q(.)$ and (c) holds by inserting $\lambda_t = 1/(2h_1^2 P_s + \sigma^2)$ and $\lambda_y = 1/\sigma^2$. Hence, the overall average probability of error can be approximated as

$$\begin{aligned} &P_e \leq P_{ML}^2 + 2P_{ML} \Pr\{\beta_2 > \beta_1\} \\ &\leq P_{ML}^2 + 2P_{ML}\, 2^{-n\left(\frac{\log_2(e)}{2} - \frac{\log_2(e)(1+2\Gamma)^2}{1+(1+2\Gamma)^4}\right)} \\ &\leq 2^{-n \min\left\{2E_r(R), E_r(R) + \frac{\log_2(e)}{2} - \frac{\log_2(e)(1+2\Gamma)^2}{1+(1+2\Gamma)^4} - \frac{2}{n}\right\}} \end{aligned}$$

where we use $P_{ML} = \exp\{-nE_r(R)\}$ as the standard ML error probability [8] at each RS. From the definition (3), as $n \to \infty$, the corresponding error exponent is given by

$$E_{DF}(R) = \min\left\{2E_r(R), E_r(R) + \frac{\log_2(e)}{2} - \frac{\log_2(e)(1+2\Gamma)^2}{1+(1+2\Gamma)^4}\right\}, \tag{14}$$

which indicates that by the proposed DF relaying allowing multiple RSs (here two) to participate in communications between the source and the destination always provides *diversity gains* (against noise) at all SNR ranges.

B. CF relaying with Gaussian Signaling

For CF relaying, assuming phase compensation and Gaussian mapping at the RSs, the quantizer outputs are given, in vector form, by

$$\mathbf{v} = \mathbf{y}_R + \mathbf{z}_q = \mathbf{h}\, x + \mathbf{z} + \mathbf{z}_q$$

where $\mathbf{h} = [h_1\ h_2]^T$, $\mathbf{z}, \mathbf{z}_q \in \mathbb{C}^{2\times 1}$ and $z_k \sim \mathcal{CN}(0, \sigma^2)$ and $z_{q,k} \sim \mathcal{CN}(0, D_k)$ for $k = 1, 2$. Define the 2×2 matrix $\mathbf{W} = \text{diag}\{\sigma^2 + D_1, \sigma^2 + D_2\}$. Then, $E_0(\rho, P_s)$ becomes

$$\begin{aligned} E_0(\rho, P_s) &= \rho \log_2 \left|\mathbf{I} + \frac{P_s}{1+\rho}\mathbf{W}^{-1}\mathbf{h}\mathbf{h}^H\right| \\ &= \rho \log_2 \left(1 + \frac{P_s}{1+\rho}\left[\frac{h_1^2}{\sigma^2 + D_1} + \frac{h_2^2}{\sigma^2 + D_2}\right]\right). \end{aligned} \tag{15}$$

[3]$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ is the standard tail function for Gaussian RVs.

As in the process of achievable rate calculation, we have the following compression rate constraints [7]:

$$\log_2\left(\frac{\sigma^2_{v_k}}{D_k}(1-\zeta^2)\right) \le C_k \qquad k=1,2,$$
$$\log_2\left(\frac{\sigma^2_{v_1}}{D_1}\frac{\sigma^2_{v_2}}{D_2}(1-\zeta^2)\right) \le C_1+C_2 \tag{16}$$

where $\sigma^2_{v_k}=h_k^2P_s+\sigma^2+D_k$, $k=1,2$, and $\zeta\in[-1,1]$ is the correlation factor between v_1 and v_2.

Then, the random coding EE corresponding to the CF relaying scheme is given by

$$E_{r,CF}(R)=\max_{0\le\rho\le1}\left[E_0(\rho,P_s)-\rho R\right] \tag{17}$$

subject to the rate constraints specified above. We note that $E_{r,CF}(R)$ is a decreasing function of both D_1 and D_2, hence the minimum possible distortion values will result in optimum error exponent.

C. QF relaying with Non-Gaussian Signaling

In this section, we examine the EE for PRNs where the source transmits (n,R) block code where each letter of each codeword is independently drawn according to a probability distribution $p(x)$ and an M-QAM constellation is used where 2^{nR} messages (alphabet size) are encoded over a block of n symbols. The received signals at the RSs are simply quantized by using uSQ, where correlation information is discarded. We assume that each symbol $x=(x^R,x^I)=x^R+jx^I$ on the M-QAM constellation has equal probability $p(x)=1/M$, and $p(x^R)=p(x^I)=1/\sqrt{M}$.

The input-output model (1) can be decomposed into real and imaginary parts as follows

$$\underline{y}_{R_k}=\begin{bmatrix} y^R_{R_k}\\ y^I_{R_k}\end{bmatrix}=\begin{bmatrix}\Re\{y_{R_k}\}\\ \Im\{y_{R_k}\}\end{bmatrix}=\begin{bmatrix}h_kx^R+z^R_k\\ h_kx^I+z^I_k\end{bmatrix}, \tag{18}$$

where $x^R=\Re\{x\}$ and $x^I=\Im\{x\}$ are the real and imaginary parts of the signal transmitted from the source, respectively, and $\mathbb{E}[(X^R)^2]=\mathbb{E}[(X^I)^2]=\frac{P_s}{2}$ (note that $\mathbb{E}[X^RX^I]=0$). Noise components have zero mean and covariance matrix $\mathbb{E}[(Z^R_k)^2]=\mathbb{E}[(Z^I_k)^2]=\frac{\sigma^2}{2}$.

The uSQ process at each RS follows the same steps as in [9]. Then, for a given source input signal x, the probability that the quantizer output is in the $\underline{l}=(l^R,l^I)$-th quantizing interval, i.e., $\underline{V}_k=(V^R_k,V^I_k)=\underline{v}_{k,\underline{l}}=(v^R_{k,l^R},v^I_{k,l^I})$, $k=1,2$, is given by

$$\begin{aligned}\Pr\left[\underline{V}_k=\underline{v}_{k,\underline{l}}|x\right]&=\Pr\left[(V^R_k,V^I_k)=(v^R_{k,l^R},v^I_{k,l^I})|x\right]\\&=\Pr\left[V^R_k=v^R_{k,l^R}\mid x^R\right]\Pr\left[V^I_k=v^I_{k,l^I}|x^I\right]\\&=\Pr\left[y^R_{R_k}\in\mathcal{S}^R_{k,l^R}|x^R\right]\Pr\left[y^I_{R_k}\in\mathcal{S}^I_{k,l^I}|x^I\right]\end{aligned} \tag{19}$$

where

$$\Pr\left[y^R_{R_k}\in\mathcal{S}^R_{k,l^R}|x^R\right]=Q\left(\frac{u^R_{k,l^R}-h_kx^R}{\sigma/\sqrt{2}}\right)-Q\left(\frac{u^R_{k,l^R+1}-h_kx^R}{\sigma/\sqrt{2}}\right)$$
$$\Pr\left[y^I_{R_k}\in\mathcal{S}^I_{k,l^I}|x^I\right]=Q\left(\frac{u^I_{k,l^I}-h_kx^I}{\sigma/\sqrt{2}}\right)-Q\left(\frac{u^I_{k,l^I+1}-h_kx^I}{\sigma/\sqrt{2}}\right)$$

for $\underline{l}=[1,\dots,L^R_k]\times[1,\dots,L^I_k]$ where L^R_k and L^I_k denote the number of quantization outputs for real and imaginary parts of the received signal at the k-th relay, $k=1,2$.

We note that for symmetric channel gains $h=h_1=h_2$ and $\sqrt{L_k}=L^R_k=L^I_k=2^{\frac{C_k}{2}}$, the quantization steps for both real and imaginary parts become symmetric, then the representation points and the transition levels become the same, i.e, $v^R_{k,l}=v^I_{k,l}=\hat{v}_{k,l}$ and $u^R_{k,l}=u^I_{k,l}=\hat{u}_{k,l}$ for $l=1,\dots,L_k$.

The destination performs ML decoding based on the observations v_1,v_2, which are the representation points corresponding to the received signals at each RS. Then, we have the following EE for the QF relaying with M-QAM at the source and uSQ at the RSs

$$E_{r,QF}(R)=\max_{0\le\rho\le1}\left[E_0(\rho,p(x)=1/M)-\rho R\right], \tag{20}$$

where $E_0(\rho,p(x)=1/M)=E_0(\rho)$ is defined as

$$\begin{aligned}E_0(\rho)&=-\ln\left[\sum_{v_1,v_2}\left[\sum_x\frac{1}{M}p(v_1,v_2|x)^{\frac{1}{1+\rho}}\right]^{1+\rho}\right]\\&=-\ln\left[\sum_{v_1,v_2}\left[\sum_x\frac{1}{M}p(v_1|x)^{\frac{1}{1+\rho}}p(v_2|x)^{\frac{1}{1+\rho}}\right]^{1+\rho}\right]\\&=E^*_0(\rho)+E^{**}_0(\rho)=2E^*_0(\rho)\end{aligned} \tag{21}$$

where

$$E^*_0(\rho)=-\ln\left[\sum_{v^R_1,v^R_2}\left[\sum_{x^R}\frac{1}{\sqrt{M}}\left[p(v^R_1|x^R)p(v^R_2|x^R)\right]^{\frac{1}{1+\rho}}\right]^{1+\rho}\right],$$
$$E^{**}_0(\rho)=-\ln\left[\sum_{v^I_1,v^I_2}\left[\sum_{x^I}\frac{1}{\sqrt{M}}\left[p(v^I_1|x^I)p(v^I_2|x^I)\right]^{\frac{1}{1+\rho}}\right]^{1+\rho}\right]$$

and $p(v^R_k|x^R)$ and $p(v^I_k|x^I)$, for $k=1,2$, are evaluated as in (19). With these settings (20) becomes

$$E_{r,QF}(R)=\max_{0\le\rho\le1}\left[2E^*_0(\rho)-\rho R\right]. \tag{22}$$

IV. Numerical Results

We compare the random coding EE performances of the relaying strategies studied above for the symmetric system model case where the channel gains from the source to RSs are the same, i.e., $h=h_1=h_2=1$, and the link capacities from the RSs to the destination are the same, $C=C_1=C_2$.

In Fig. 2 and Fig. 3, we plot the EEs given by (14), (17) and (22) corresponding to DF, CF and QF (with 4-QAM at the source and uSQ at the RSs) relaying strategies with respect to transmission rate R [bits/transmission] for fixed $\Gamma=\frac{P_sh^2}{\sigma^2}=\{0,10\}$ [dB]. In Fig. 2, which corresponds to a low SNR regime, we see that at rates above 0.2 [bits/transmission] the proposed simple and practical QF relaying has better EE than both DF and CF. However, when we operate at rates lower than 0.2 [bits/transmission], the EE for the proposed DF relaying strategy outperforms the others. In Fig. 3, which corresponds to a high SNR regime, we see that at all rates our proposed

QF relaying performs the worst, which could be explained as follows: since the backhaul rate is fixed whilst the SNR is increased the proposed QF strategy cannot fully exploit the structure of the modulation scheme used at the source. From this plot we can also see that the achieved EE with the proposed DF relaying is better than that of the CF relaying strategy at low to moderate rates.

V. Conclusions

In this paper, we studied the PRN consisting of a single source and two relays which are connected to a destination via an error-free finite capacity backhaul. We evaluated the random coding EEs corresponding to DF, CF and QF relaying strategies for the PRN in order to have thorough characterization of system performance. Moreover, through numerical analysis we illustrated that the EEs achieved by using QF relaying along with non-Gaussian signaling (M-QAM) at the source and symbol-by-symbol uSQs at the relays is better than that achieved by DF and CF relaying strategies when the system is in the low SNR regime and the backhaul capacity is large enough. Using a finite constellation, such as M-QAM, at the source node along with simple processing, such as the proposed QF scheme, at the relay can provide better EEs compared to more complex schemes. This is due to the structure inherent in the considered modulation scheme, which Gaussian signaling lacks.

Appendix A
The pdf of the difference of exponentially distributed RVs

Let T and Y be two independent exponentially distributed RVs with respective means $\mathbb{E}[T] = \lambda_t$ and $\mathbb{E}[Y] = \lambda_y$. Now define a new RV, $Z = T - Y$.

We want to find the pdf of Z. The cdf of Z is given by

$$F_Z(z) = P(Z \leq z) = P(T - Y \leq z)$$
$$= \begin{cases} \int_0^{\infty}\int_0^{z+y} f(t)f(y)dtdy = 1 - \dfrac{\lambda_y}{\lambda_y+\lambda_t}e^{-\lambda_t\, z}, & z \geq 0 \\ \int_{-z}^{\infty}\int_0^{z+y} f(t)f(y)dtdy = \dfrac{\lambda_t}{\lambda_y+\lambda_t}e^{\lambda_y\, z}, & z < 0. \end{cases}$$

Then, the pdf of Z is calculated as

$$f_Z(z) = \frac{\partial F_Z(z)}{dz} = \begin{cases} \dfrac{\lambda_t\, \lambda_y}{\lambda_t+\lambda_y}e^{-\lambda_t\, z} & , z \geq 0 \\ \dfrac{\lambda_t\, \lambda_y}{\lambda_t+\lambda_y}e^{\lambda_y\, z} & , z < 0 \end{cases} \qquad (23)$$

Then, the mean and variance of Z are given by

$$\mu_Z = \mathbb{E}[Z] = \int_{-\infty}^{\infty} z f(z) dz = \frac{1}{\lambda_t} - \frac{1}{\lambda_y},$$
$$\sigma_Z^2 = \mathbb{VAR}[Z] = \frac{1}{(\lambda_t+\lambda_y)^2}\left[\frac{\lambda_y^2}{\lambda_t^2} + \frac{\lambda_t^2}{\lambda_y^2}\right]. \qquad (24)$$

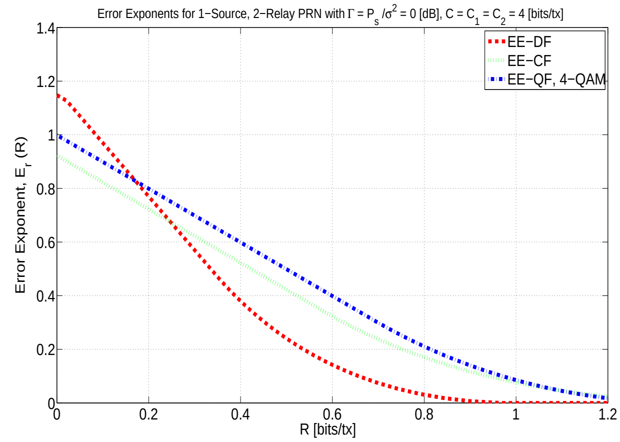

Fig. 2. Error exponents for 1-Source, 2-Relay PRN with $\Gamma = \frac{P_s h^2}{\sigma^2} = 0$ [dB] and $C = C_1 = C_2 = 4$[bits/transmission].

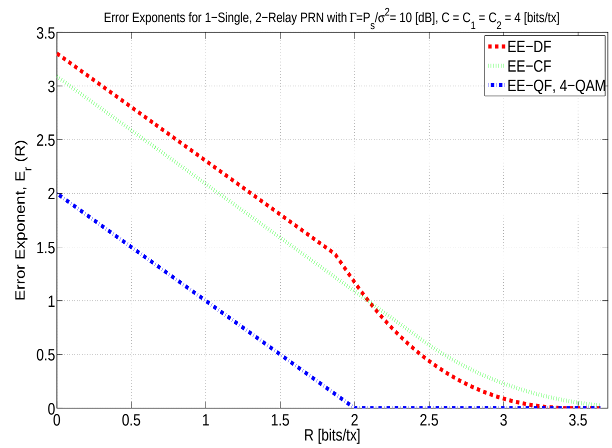

Fig. 3. Error exponents for 1-Source, 2-Relay PRN with $\Gamma = \frac{P_s h^2}{\sigma^2} = 10$ [dB] and $C = C_1 = C_2 = 4$[bits/transmission].

References

[1] J. N. Laneman, D. N. C. Tse, and G.W. Wornell, ”Cooperative diversity in wireless networks: Efficient protocols and outage behavior,” *IEEE Trans. Info. Theory*, vol. 50, no. 12, pp. 30623080, Dec. 2004.

[2] A. Host-Madsen and J. Zhang, ” Capacity bounds and power allocation for wireless relay channel,” *IEEE Trans. Inform. Theory*, vol. 51, no. 6, pp. 20202040, June 2005.

[3] G. Kramer, M. Gastpar, and P. Gupta, ”Cooperative strategies and capacity theorems for relay networks,” *IEEE Trans. Info. Theory*, 2004.

[4] S. Shamai (Shitz), O. Simeone, O. Somekh, and H. V. Poor, ”Joint multi-cell processing for downlink with limited-capacity backhaul,” *in the proc. of ITA Workshop*, Jan. 27- Feb. 1, San Diego, CA, 2008.

[5] A. Sanderovich, S. Shamai (Shitz), Y. Steinberg, and G. Kramer, ”Communication via decentralized processing,” *IEEE Trans. Info. Theory*, vol. 54, no. 7, pp. 3008-3023, Jul. 2008.

[6] B. E. Schein, ”Distributed Coordination in network information theory,” *PhD. dissertation*, MIT, Cambridge, MA, Oct. 2001.

[7] T. M. Cover and A. E. Gamal, ”Capacity theorems for the relay channel,” *IEEE Trans. Info. Theory*, vol. 25, pp. 572584, Sep. 1979.

[8] R. G. Gallager, ”Information Theory and Reliable Communication,” Wiley, New York, 1968.

[9] A. Gersho and R. Gray, ”Vector Quantization and Signal Compression,” Boston, MA, Kluwer Academic Press, 1991.