

Maximizing the spectral efficiency of LDPC-encoded CDMA

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Abstract

We investigate the spectral efficiency achievable by random synchronous CDMA with QPSK modulation and binary error-control codes, in the large system limit where the number of users, the spreading factor and the code block length go to infinity. For given LDPC code ensembles, we maximize spectral efficiency assuming an MMSE successive stripping decoder for the cases of equal rate and equal power users. In both cases, the maximization of spectral efficiency can be formulated as a linear program and admits a simple closed-form solution that can be readily interpreted in terms of power and rate control.

Keywords: Low-Density Parity-Check codes, channel capacity, multiuser detection.

1 Introduction

All points in the capacity region of the scalar Gaussian multiple-access channel are achievable by successive single-user encoding decoding and interference cancellation (stripping) [1, 2]. The Cover-Wyner capacity region was generalized in [3] to encompass the case of CDMA with arbitrary signature waveforms. As in the scalar case, all points in the boundary of the capacity region of the CDMA channel can also be achieved by stripping and single-user encoding/decoding, as long as the stripping decoders incorporate MMSE filters against yet undecoded users at each successive cancellation stage [4]. Key to the optimality of stripping is the use of Gaussian codes of rate arbitrarily close (but not larger) than the capacity of the channel obtained by removing the already decoded users. In this way, optimal spectral efficiency is achieved by simple single-user coding and decoding, with linear complexity in the number of users.

For given signature waveforms, successive stripping generally requires that every user must transmit at a different rate, or must be received at a different SNR level. This can be avoided by designing the signature waveforms such that the equal-rate point coincides with a vertex of the equal-power capacity region. However, optimizing the signature waveforms (e.g., [5, 6]) is highly impractical in real-life applications, where transmission is usually affected by frequency selective fading channels.

On the other hand, existing nonorthogonal CDMA systems [7, 8] are largely based on pseudo-random waveforms. The maximum spectral efficiency of randomly spread (synchronous) CDMA, in the large system limit, where the number of users and the spreading factor grow without bound while their ratio tends to a constant β , was found in the case of power-constrained inputs in [9, 10], and in the case of binary antipodal inputs in [11].

While in the power-constrained case capacity is achieved by Gaussian inputs, practical systems make use of discrete small-size modulation alphabets. Given its widespread application and the fact that QPSK is optimal in the wideband low SNR regime [12] we shall restrict our analysis to QPSK-modulated CDMA.

We consider a pragmatic approach to QPSK-modulated CDMA based on applying single-user binary coding and the same stripping decoding approach which would be optimal for Gaussian codes. Moreover, we constrain our system to have only a finite number of coding rates and/or of received SNR levels. For this setting, we compute the achievable spectral efficiency in the large system regime with optimal (i.e., single-user capacity achieving) binary codes and with the best known LDPC code ensembles [13], in the limit for large code block length, in the cases of equal received SNRs or equal rate users. The proposed equal-power and equal-rate design approaches can be effectively applied to non-asymptotic code block length, and provide a simple tool to dimension CDMA systems for given target BER, user codes, and desired spectral efficiency.

The rest of the paper is organized as follows. Section 2 presents the basic synchronous CDMA AWGN model where users are grouped into a finite number of classes such that users in a given class have the same rate and received SNR. The existing results on the optimum spectral efficiency of the power-constrained CDMA channel are summarized in Section 3 with particular emphasis on the finite class model. Our choice for the input constellation is QPSK, justified on the basis of complexity and asymptotic optimality, as shown in Section 4.1. Then, we consider the optimization of the received power profile of the different classes in Section 4.2, assuming that the users employ equal-rate codes. Conversely, in Section 4.5 we optimize the code rate profile assuming equal received SNR for all users. Both problems are formulated as linear programs whose solution can be found explicitly. Section 5 presents numerical examples of both methods when the user codes are irregular LDPC codes found in [13].

2 Synchronous CDMA canonical model

We consider the complex baseband discrete-time channel model

$$\mathbf{y}_i = \mathbf{S}\mathbf{x}_i + \mathbf{n}_i, \quad i = 1, \dots, n \quad (1)$$

originated by sampling at the chip-rate a synchronous CDMA system [14], where:

- 1) $\mathbf{y}_i, \mathbf{n}_i \in \mathbb{C}^N$, are the vector of received chip-rate samples and the corresponding AWGN samples $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$ received at time i ;

- 2) $\mathbf{S} \in \mathbb{C}^{N \times K}$ contains the user spreading sequences by columns. Spreading sequences are proper complex, known to the receiver, with i.i.d. chips with zero mean, variance $1/N$ and finite fourth order moment;
- 3) $\mathbf{x}_i \in \mathbb{C}^K$ is the vector of user modulation symbols transmitted at time i , where its k -th component $x_{k,i}$ takes on values in some signal constellation, with given average energy per symbol $E[|x_{i,k}|^2]$, possibly different for each user;
- 4) N, K and n denote the spreading factor, the number of users and the code block length, respectively.

For the purpose of system design it is convenient to consider a system formed by J user classes. The size of class j is K_j , and we denote by $\beta_j = K_j/N$ the “class load” of class j . Thus, the total channel load is

$$\beta = \sum_{j=1}^J \beta_j \quad \text{users/chip.}$$

Users in class j have the same received SNR, denoted by γ_j . Without loss of generality, we assume $\gamma_1 \leq \dots \leq \gamma_J$. The total system spectral efficiency is given by

$$\rho = \sum_{j=1}^J \beta_j R_j$$

where R_j denotes the average rate of users in class j . The users individual E_b/N_0 's are in general different. Nevertheless, for the sake of comparison with a reference equal-rate equal-power system, it is convenient to define a “system” E_b/N_0 by

$$\left(\frac{E_b}{N_0}\right)_{\text{sys}} \triangleq \frac{\sum_{j=1}^J \beta_j \gamma_j}{\sum_{j=1}^J \beta_j R_j} \quad (2)$$

which coincides with the individual E_b/N_0 's in the case where users are dynamically assigned to the classes so that each user belongs to class j for a fraction β_j/β of the time.

3 Existing results on fundamental limits

In [10] the spectral efficiency (in bit/s/Hz) of random CDMA in the large system limit ($K, N \rightarrow \infty$ with $K/N = \beta$) subject to fading with an input power constraint is found to be

$$\mathcal{C}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \mathcal{C}^{\text{mmse}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) + \log_2 \frac{1}{\eta} + (\eta - 1) \log_2 e \quad (3)$$

where $\boldsymbol{\beta} \triangleq (\beta_1, \dots, \beta_J)$ and $\boldsymbol{\gamma} \triangleq (\gamma_1, \dots, \gamma_J)$, η is the large-system multiuser efficiency [14] of the linear MMSE receiver, given by the solution of the Tse-Hanly equation [15], which for later use we write as

$$\eta = f_J(\eta, \boldsymbol{\beta}_J) \quad (4)$$

where we define

$$f_j(\eta, z) \triangleq \left(1 + z \frac{\gamma_j}{1 + \gamma_j \eta} + \sum_{i=1}^{j-1} \beta_i \frac{\gamma_i}{1 + \gamma_i \eta} \right)^{-1} \quad (5)$$

and where $C^{\text{mmse}}(\boldsymbol{\beta}, \boldsymbol{\gamma})$ is the achievable spectral efficiency of a system based on linear MMSE filtering followed by single-user decoding, given by

$$C^{\text{mmse}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{j=1}^J \beta_j \log_2(1 + \gamma_j \eta) \quad (6)$$

The spectral efficiencies in (3) and in (6) are achieved with codes whose empirical distributions are Gaussian.

As shown in [10], the supremum of (3) over all possible $J, \boldsymbol{\beta}, \boldsymbol{\gamma}$ (for a fixed E_b/N_0 and β) is achieved by $J = 1$ (one class only). This can be readily seen by writing the total spectral efficiency for finite K, N and given \mathbf{S} as $\frac{1}{N} \log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \mathbf{A}^2 \mathbf{S}^H)$, where the k -th user SNR is denoted by $\text{SNR} |\mathbf{A}_k|^2$, where $\frac{1}{K} \sum_{k=1}^K |\mathbf{A}_k|^2 = 1$ and $\mathbf{A} \triangleq \text{diag}(|\mathbf{A}_1|, \dots, |\mathbf{A}_K|)$. Then, we notice that, for $K/N = \beta$ and assuming that the empirical distribution of the scaling factors $|\mathbf{A}_k|$ converges to any fixed (non-random) distribution, the limit

$$\lim_{K \rightarrow \infty} \frac{1}{N} \log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \mathbf{A}^2 \mathbf{S}^H) = \lim_{K \rightarrow \infty} \frac{1}{N} E [\log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \mathbf{A}^2 \mathbf{S}^H)]$$

holds with probability 1 [10]. Finally, by averaging over all $K \times K$ permutation matrices $\boldsymbol{\Pi}$, by noticing that \mathbf{S} and $\mathbf{S} \boldsymbol{\Pi}$ are identically distributed and by using Jensen's inequality, it follows from the concavity of $\log \det(\cdot)$ on the cone of non-negative definite Hermitian symmetric matrices that

$$\begin{aligned} E [\log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \mathbf{A}^2 \mathbf{S}^H)] &= \frac{1}{K!} \sum_{\boldsymbol{\Pi}} E [\log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \boldsymbol{\Pi} \mathbf{A}^2 \boldsymbol{\Pi}^H \mathbf{S}^H)] \\ &\leq E \left[\log_2 \det \left(\mathbf{I} + \frac{\text{SNR}}{K!} \sum_{\boldsymbol{\Pi}} \mathbf{S} \boldsymbol{\Pi} \mathbf{A}^2 \boldsymbol{\Pi}^H \mathbf{S}^H \right) \right] \\ &= E [\log_2 \det(\mathbf{I} + \text{SNR} \mathbf{S} \mathbf{S}^H)] \end{aligned}$$

where the last line is achieved when all users are received with the same SNR.

The supremum over β is achieved for $\beta \rightarrow \infty$, and coincides with the AWGN single-user capacity C^* , implicitly given by

$$\frac{2^{C^*} - 1}{C^*} = \frac{E_b}{N_0} \quad (7)$$

The spectral efficiency $C(\boldsymbol{\beta}, \boldsymbol{\gamma})$ can be achieved by single-user decoding with successive stripping and MMSE filtering against undemodulated users. Suppose that users are decoded one by one, starting from users in class J , then class $J - 1$ and so on. Then, $C(\boldsymbol{\beta}, \boldsymbol{\gamma})$

can be written as

$$C(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{j=1}^J \int_0^{\beta_j} \log_2(1 + \gamma_j \eta_j(z)) dz \quad (8)$$

where $\eta_j(z)$ is the solution to $\eta = f_j(\eta, z)$. Notice that stripping of the users one by one implies that users in the same class have different rates. Namely, the user decoded in position $\lfloor K_j z / \beta_j \rfloor$ of class j (where $z \in [0, \beta_j]$), transmits at rate $\log_2(1 + \gamma_j \eta_j(z))$.

4 Approaching the optimal spectral efficiency

4.1 QPSK input constellations

Information theory teaches us that one way to approach (3) is to use single-user capacity approaching codes for the AWGN channel, successive interference cancellation and MMSE filtering at each cancellation stage [4]. Furthermore, substantial progress has been made in the last few years in designing binary codes and low-complexity decoders whose rate comes fairly close to single-user capacity at vanishing BER. Among those modern codes are Turbo codes, Repeat-Accumulate (RA) codes, and Low-Density Parity-Check codes (LDPC), all of which are decoded by efficient iterative techniques (see the special issue [16] and references therein). These code ensembles are characterized by rate-threshold pairs (R, g) , such that for $\text{SNR} \geq g$ the BER can be made arbitrarily small in the limit of $n \rightarrow \infty$. For carefully optimized code ensembles [17, 18, 19, 20], on the standard single-user additive white Gaussian noise channel, the rate-threshold pairs achieved so far come remarkably close to the curve $R = C_{\text{qpsk}}(\text{SNR})$, where

$$C_{\text{qpsk}}(\text{SNR}) = 2 \left(1 - \int_{-\infty}^{\infty} \log_2 \left(1 + e^{-2} \text{SNR}^{-2} \sqrt{\text{SNR} v} \right) \frac{e^{-v^2/2}}{\sqrt{2\pi}} dv \right) \quad (9)$$

is the QPSK-input AWGN channel capacity, as a function of SNR. For example, Fig. 1 shows the QPSK capacity (9) (solid curve) and rate-threshold pairs (marks) corresponding to some LDPC code ensembles from [13].

Then, it makes sense to design CDMA systems assuming that decoding is error-free when the decoder operates above its threshold SNR. Our goal is to find the vectors $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ so that, at each stripping decoder stage, the threshold requirement of each single-user decoder is satisfied. We shall refer to this condition as the *successive decodability condition*.

In the large system limit, under our system assumptions, it is well-known that the residual interference at the output of the MMSE filter at any cancellation stage is complex Gaussian with circular symmetry [21]. Assuming optimal QPSK codes characterized by the rate-threshold pairs $(R, C_{\text{qpsk}}^{-1}(R))$, for $R \in [0, 2]$, (see Fig. 1), the spectral efficiency achieved by a stripping decoder is given by

$$C_{\text{qpsk}}(\boldsymbol{\beta}, \boldsymbol{\gamma}) = \sum_{j=1}^J \int_0^{\beta_j} C_{\text{qpsk}}(\gamma_j \eta_j(z)) dz \quad (10)$$

Fig. 2 shows $C_{\text{qpsk}}(\beta, \gamma)$ and $C(\beta, \gamma)$ (for a single-class system, i.e., $J = 1$) vs. β , for $E_b/N_0 = 3$ and 10 dB. The corresponding AWGN capacity C^* is shown for comparison. We notice that the loss incurred by QPSK codes with respect to Gaussian codes gets more pronounced as E_b/N_0 increases. Although for any fixed E_b/N_0 and sufficiently large β , the loss vanishes, for high E_b/N_0 exceedingly large values of β are required to make the loss negligible.

The following result shows that as the system load grows without bound, QPSK suffers no loss of optimality. The result applies to the general case where the received user SNRs are given by $\text{SNR}|A_k|^2$, where the scaling factors $|A_k|$ have been defined in Section 3, under the mild requirement that, as $K \rightarrow \infty$, the empirical distribution of the $\{|A_k|\}$ converges to a given non-random distribution $F_{|A|}$. For example, as in [10] $|A_k|^2$ might represent the fading coefficient of user k .

Theorem 1. Let

$$C_Q(\beta, \text{SNR}) = \int_0^\beta E[C_{\text{qpsk}}(|A|^2 \text{SNR} \eta(z, \text{SNR}))] dz \quad (11)$$

where $\eta(z, \text{SNR})$ is the solution to

$$\eta + zE \left[\frac{\text{SNR}|A|^2 \eta}{1 + \text{SNR}|A|^2 \eta} \right] = 1 \quad (12)$$

where $|A| \sim F_{|A|}$.

Fix β and $\frac{E_b}{N_0}$ and define

$$C_Q(\beta, \frac{E_b}{N_0}) = C_Q(\beta, \text{SNR}) \quad (13)$$

for the SNR satisfying

$$\frac{E_b}{N_0} C_Q(\beta, \text{SNR}) = \beta \text{SNR} \quad (14)$$

Then, for all $\frac{E_b}{N_0} \geq \log_e 2$,

$$\lim_{\beta \rightarrow \infty} C_Q(\beta, \frac{E_b}{N_0}) = C^* \quad (15)$$

Proof. Fix $\frac{E_b}{N_0} > \log_e 2$. In view of the result shown in [10, Eq. 163] and since the use of QPSK cannot improve upon the result obtained with Gaussian inputs with the same power, it is enough to show that

$$\lim_{\beta \rightarrow \infty} C_Q(\beta, \frac{E_b}{N_0}) \geq C^* \quad (16)$$

where C^* is given by (7).

To show (16), we will show that for every β and SNR

$$C_Q(\beta, \text{SNR}) \geq \log_2(1 + \beta \text{SNR}) - \frac{\kappa(|\mathbf{A}|)\beta \text{SNR}^2}{\beta \text{SNR} + 1} \log_2 e, \quad (17)$$

where

$$\kappa(|\mathbf{A}|) = \frac{E[|\mathbf{A}|^4]}{(E[|\mathbf{A}|^2])^2}$$

denotes the kurtosis of the distribution of $|\mathbf{A}|$.

The bound in (17) will be sufficient for our purposes because if we choose the following signal-to-noise ratio

$$\text{SNR}_\beta = \frac{C^* E_b}{\beta N_0} \quad (18)$$

then,

$$C_Q(\beta, \text{SNR}_\beta) \geq \log_2(1 + \beta \text{SNR}_\beta) - \kappa(|\mathbf{A}|) \text{SNR}_\beta \frac{\frac{E_b}{N_0} C^*}{\frac{E_b}{N_0} C^* + 1} \log_2 e. \quad (19)$$

$$\rightarrow \log_2\left(1 + \frac{E_b}{N_0} C^*\right) \quad (20)$$

$$= C^* \quad (21)$$

Furthermore the $\frac{E_b}{N_0}$ required by SNR_β is upper bounded by

$$\frac{\beta \text{SNR}_\beta}{C_Q(\beta, \text{SNR}_\beta)} \leq \frac{\frac{E_b}{N_0} C^*}{\log_2\left(1 + \frac{E_b}{N_0} C^*\right) - \epsilon C^*} \quad (22)$$

$$= \frac{E_b}{N_0} \frac{1}{1 - \epsilon} \quad (23)$$

for an arbitrarily small ϵ , provided β is large enough.

To show (17) we need the following two inequalities:

$$C_{\text{qpsk}}(x) \geq (x - x^2) \log_e 2 \quad (24)$$

and

$$\eta \geq \frac{1}{1 + \beta \text{SNR}} \quad (25)$$

where η is the solution to

$$\eta + \beta E \left[\frac{\text{SNR} |\mathbf{A}|^2 \eta}{1 + \text{SNR} |\mathbf{A}|^2 \eta} \right] = 1 \quad (26)$$

To show (25) rewrite the Tse-Hanly equation as

$$\text{SNR} = \eta \text{SNR} + \beta \text{SNR} E \left[\frac{\text{SNR} |\mathbf{A}|^2 \eta}{1 + \text{SNR} |\mathbf{A}|^2 \eta} \right] \quad (27)$$

$$\leq \eta \text{SNR} + \beta \text{SNR}^2 \eta \quad (28)$$

where the inequality comes from $E[|A|^2] = 1$. Inequality (24) follows from the definition of C_{qpsk} .

Now, (17) readily follows from

$$C_Q(\beta, \text{SNR}) = E[C_{\text{qpsk}}(|A|^2 \text{SNR} \eta(z, \text{SNR}))] dz \quad (29)$$

$$\geq \int_0^\beta E \left[C_{\text{qpsk}} \left(\frac{|A|^2 \text{SNR}}{1 + z \text{SNR}} \right) \right] dz \quad (30)$$

$$\geq \int_0^\beta E \left[\frac{|A|^2 \text{SNR}}{1 + z \text{SNR}} \log_2 e - E \left[\frac{|A|^4 \text{SNR}^2}{(1 + z \text{SNR})^2} \right] \log_2 e \right] dz \quad (31)$$

$$\geq \int_0^\beta \frac{\text{SNR}}{1 + z \text{SNR}} \log_2 e - \frac{\kappa(|A|) \text{SNR}^2}{(1 + z \text{SNR})^2} \log_2 e dz \quad (32)$$

$$= \log_2(1 + \beta \text{SNR}) - \frac{\kappa(|A|) \beta \text{SNR}^2}{1 + \beta \text{SNR}} \log_2 e \quad (33)$$

thus concluding the proof. \square

Interestingly, the optimality of QPSK in the large β limit proved in Theorem 1 is different in nature from its wideband optimality proved in [12]. In fact, as a consequence of the rotational invariance of the spreading sequences, Theorem 1 also holds if the modulation is BPSK. A related result on the optimality of binary inputs in the absence of spreading admits a very different (central-limit based) proof [3].

In the following we consider two alternative *pragmatic* CDMA optimization problems: (1) equal-rate, non-uniform power, and (2) equal-power, non-uniform rate systems. We assume that, in both cases, users in each class i are decoded in parallel by a bank of single-user decoders, while classes are stripped off from J to 1, i.e., in decreasing SNR order (for the equal-rate case) or in increasing rate order (for the equal-power case). Notice that our approach is pragmatic in two ways: it makes use of QPSK rather than Gaussian codes and it performs class-by-class stripping, rather than user-by-user, as implied by expressions (8) and (10).

4.2 Optimization for equal-rate systems

We assume that users in all classes make use of codes drawn randomly and independently from the same ensemble with rate-threshold pair (R, g) . The Signal to Interference plus Noise Ratio (SINR) at the output of the MMSE filter for class i users, assuming that all users in classes $i + 1, \dots, J$ have been perfectly canceled, is given by $\gamma_i \eta_i(\beta_i)$. Hence, the condition for successive decodability of all users is

$$\eta_i(\beta_i) \geq g/\gamma_i, \text{ for all } i = 1, \dots, J.$$

We fix the received power levels γ , and consider the optimization of the class loads β . Without loss of generality, we assume $\gamma_1 \geq g$, since for all j such that $\gamma_j < g$, we would have trivially $\beta_j = 0$. This problem can be formulated as a linear program as follows.

Because of the monotonicity in the first argument of the function in (5), and the fact that the solution to $\eta_j(z) = f_j(\eta_j(z), z)$, is unique [15] we can conclude that

$$\forall x \in [0, \infty) \quad x \leq \eta_j(z) \Leftrightarrow x \leq f_j(x, z) \quad (34)$$

Accordingly, the successive decodability condition is equivalent to

$$\left(1 + \sum_{j=1}^i \beta_j \frac{\gamma_j}{1 + \gamma_j \frac{g}{\gamma_i}}\right)^{-1} \geq \frac{g}{\gamma_i}, \quad \forall i = 1, \dots, J \quad (35)$$

which can be written in compact form as

$$\mathbf{A}\boldsymbol{\beta} \leq \mathbf{b},$$

where \mathbf{A} is a $J \times J$ lower triangular matrix with non-zero elements

$$a_{i,j} = \frac{(1+g)\gamma_j}{\gamma_i + \gamma_j g} \in (0, 1] \quad (36)$$

and \mathbf{b} is a positive vector with elements

$$b_i = \frac{(1+g)(\gamma_i - g)}{\gamma_i g} \quad (37)$$

Notice that $a_{i,i} = 1$, $a_{i,j}$ (for $1 \leq j \leq i$) is increasing with j and decreasing with i and b_i is increasing with i .

For a desired spectral efficiency $\rho = \beta R$, the optimal vector $\boldsymbol{\beta}$ which achieves (if possible) arbitrarily small BER with minimal $(E_b/N_0)_{\text{sys}}$ is the solution to the linear program:

$$\begin{cases} \text{minimize} & \sum_{i=1}^J \beta_i \gamma_i \\ \text{subject to} & \mathbf{A}\boldsymbol{\beta} \leq \mathbf{b} \\ & \sum_{i=1}^J \beta_i \geq \beta, \\ & \boldsymbol{\beta} \geq \mathbf{0} \end{cases} \quad (38)$$

given by the following result:

Proposition 1. The equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution with nonnegative components $\boldsymbol{\tau}$. Furthermore, the feasible set in (38) is nonempty if and only if $\beta \leq \sum_{j=1}^J \tau_j$. The solution of (38) is given explicitly by

$$\beta_i^* = \begin{cases} \tau_i, & i = 1, \dots, \hat{J} - 1 \\ \beta - \sum_{j=1}^{\hat{J}-1} \tau_j & i = \hat{J} \\ 0, & i = \hat{J} + 1, \dots, J \end{cases} \quad (39)$$

where \hat{J} denotes the minimum i for which $\beta \leq \sum_{j=1}^i \tau_j$.

Proof. See Appendix A □

4.3 Optimization for equal-power systems

In this case, we assume that the users in all classes have fixed SNR γ , but users in each class j make use of a different code ensemble, characterized by the rate-threshold pair (R_j, g_j) . Without loss of generality, we assume $R_1 \geq \dots \geq R_J$ and $g_1 \geq \dots \geq g_J$.¹ For example, the pairs (R_j, g_j) can be obtained by sampling the curve of Fig. 1 (assuming optimal binary random codes) or by taking the points corresponding to good existing LDPC code ensembles [13], also shown in Fig. 1. Without loss of generality, we assume $\gamma \geq g_1$, since for all j such that $\gamma < g_j$, we would have trivially $\beta_j = 0$.

The successive decodability condition is given by

$$\eta_i(\beta_i) \geq \frac{g_i}{\gamma} \text{ for all } i = 1, \dots, J$$

which translates into

$$\sum_{j=1}^i \beta_j \leq b_i, \quad i = 1, \dots, J \quad (40)$$

with

$$b_i = \frac{(1 + g_i)(\gamma - g_i)}{\gamma g_i} \quad (41)$$

using again property (34). Hence, for given rate-threshold pairs (R_j, g_j) , the spectral efficiency $\rho = \sum_{i=1}^J \beta_i R_i$ maximized over the class loads is obtained as the solution to the following linear program:

$$\begin{cases} \text{maximize} & \sum_{i=1}^J \beta_i R_i \\ \text{subject to} & \mathbf{L}\boldsymbol{\beta} \leq \mathbf{b} \\ & \sum_{i=1}^J \beta_i \leq \beta, \\ & \boldsymbol{\beta} \geq \mathbf{0} \end{cases} \quad (42)$$

where \mathbf{L} is a lower triangular $J \times J$ matrix with $l_{ij} = 1$ for all $i \geq j$ and where $\mathbf{b} = (b_1, \dots, b_J)^T$ with b_i given in (41). We have the following result:

Proposition 2. The problem (42) is always feasible and its solution is given explicitly by

$$\beta_i^* = \begin{cases} b_i - b_{i-1}, & i = 1, \dots, \hat{J} - 1 \\ \beta - b_{\hat{J}-1}, & i = \hat{J} \\ 0, & i = \hat{J} + 1, \dots, J \end{cases} \quad (43)$$

where $b_0 \triangleq 0$, and \hat{J} denotes the minimum i for which $\beta \leq b_i$.

¹Any good family of codes satisfies the condition that codes with larger rate have larger SNR thresholds.

Proof. See Appendix A. □

5 Numerical examples

In this section we give some examples of the equal-rate and the equal-power system designs using various good asymptotic constructions of LDPC codes found in [13].

Equal-rate design. In Fig. 3, the curves labelled by “LDPC, R=0.2, 1.0, 1.8” are the spectral efficiencies achieved by the equal-rate design with the LDPC codes of rate 0.2, 1.0 and 1.8 bit per QPSK symbol (corresponding to binary rate 0.1, 0.5 and 0.9), in the family whose rate-threshold pairs are represented by the marks in Fig. 1. The equal-rate spectral efficiency curves were obtained considering increasing values of β , and, for each β , a vector γ obtained by discretizing the interval $[g, \bar{\gamma}(\beta)]$ with step of 0.01 dB, where $\bar{\gamma}(\beta)$ is the minimum γ_J for which the feasible set of (38) is non-empty. The single-user capacity, C^* , is shown for comparison. We notice that the variable-rate design is able to approach quite closely C^* for low R , at the price of a very large load β and a large number of power levels.

Equal-power design. In Fig. 4, the curves labelled by “LDPC” and “discr.QPSK” are the spectral efficiencies achieved by the equal-power design with the LDPC code family found in [13] with rate-threshold pairs corresponding to the marks in Fig. 1, and rate-threshold pairs obtained by sampling the QPSK capacity curve from $R = 0.05$ to 1.95 with step 0.1, respectively. The fact that the “LDPC” curve does not approach the “discr.QPSK” curve at high $\frac{E_b}{N_0}$ is due to the fact that the largest rate we used from the LDPC code family of [13] was limited to 1.8 bits/channel use. It turns out that in order to approach C^* , it is necessary to have many different classes. Even a relatively finely discretized distribution of rates (such as curve “discr.QPSK”) suffers some loss away from C^* . In Fig. 4 the low $(E_b/N_0)_{\text{sys}}$ behavior of spectral efficiency is dominated by the class with lowest coding rate (and SNR threshold). In fact, the value at which spectral efficiency becomes zero is given by g_J/R_J , which is the minimum E_b/N_0 to have a vanishing fraction of users at non-zero rate.

It is worthwhile to mention that the equal-power spectral efficiency curves are obtained as the upper envelope of the solution of (42), over all $\gamma \geq g_J$ and $\beta \in [0, b_J]$, i.e., for all pairs (γ, β) for which the solution (43) exists.

6 Conclusions

We considered the optimization of a canonical coded synchronous CDMA system characterized by random spreading and QPSK signaling, in the limit for large number of users, large spreading gain, and large user code block length. Such assumptions may be regarded as “pragmatic”, in the sense that they are all motivated by today CDMA real systems.

The CDMA system considered here has low complexity, as it assumes LDPC codes and successive stripping with MMSE filters (excellent approximations to the MMSE filters can be precomputed using the large random matrix design approach of [22]). Nevertheless, the proposed optimization, driven by recent information-theoretic results, yields spectral efficiencies remarkably close to the optimal (i.e., optimizing also with respect to the user signature waveforms and using Gaussian codebooks).

We quantified the loss in spectral efficiency due to the use of QPSK *in lieu* of Gaussian inputs. The loss for high SNR is not as pronounced as in the single-user case and in fact we showed that it vanishes for large channel load β . Then, we considered two special cases of the general rate and power allocation problem: namely, the optimization of the received SNR distribution for an equal-rate system, and the optimization of the user rate distribution for an equal-power system, subject to the successive decodability condition imposed by the stripping decoder. Both problems yield linear programs that admit closed form explicit solutions.

Our numerical results show that, from a practical viewpoint, the equal-rate system design is more attractive than its equal-power counterpart since it can approach optimal spectral efficiency uniformly, for all E_b/N_0 's, provided that the individual users coding rate is small. Moreover, controlling the received user SNR is much easier and closer to today power-control schemes than allocating coding rates (and channel codes) to the users.

APPENDIX

A Proofs

Proof of Proposition 1. A necessary condition for $\boldsymbol{\beta}$ minimizing the objective function in (38) is that the constraint $\sum_j \beta_j \geq \beta$ holds with equality. Hence, without loss of generality we rewrite (38) in the canonical form

$$\begin{cases} \text{minimize} & \boldsymbol{\gamma}^T \boldsymbol{\beta} \\ \text{subject to} & -\mathbf{A}\boldsymbol{\beta} \geq -\mathbf{b} \\ & \mathbf{1}^T \boldsymbol{\beta} = \beta, \\ & \boldsymbol{\beta} \geq \mathbf{0} \end{cases} \quad (44)$$

The dual linear program is given by

$$\begin{cases} \text{maximize} & (-\mathbf{b}^T, \beta) \begin{bmatrix} \mathbf{y} \\ \alpha \end{bmatrix} \\ \text{subject to} & [-\mathbf{A}^T, \mathbf{1}] \begin{bmatrix} \mathbf{y} \\ \alpha \end{bmatrix} \leq \boldsymbol{\gamma} \\ & \mathbf{y} \geq \mathbf{0} \end{cases} \quad (45)$$

where α can be either positive or negative.

From the properties of the coefficients $a_{i,j}$ and b_j we get immediately that \mathbf{A} is invertible and the vector $\boldsymbol{\tau}$ such that $\boldsymbol{\tau} = \mathbf{A}^{-1}\mathbf{b}$ has non-negative components. The vector $\boldsymbol{\beta} \in \mathbb{R}_+^J$ maximizing $\mathbf{1}^T\boldsymbol{\beta}$ and satisfying $\mathbf{A}\boldsymbol{\beta} \leq \mathbf{b}$ is $\boldsymbol{\tau}$ (this is easily shown by contradiction, since $\boldsymbol{\tau}$ is the unique non-negative vector $\boldsymbol{\beta}$ that makes the inequality $\mathbf{A}\boldsymbol{\beta} \leq \mathbf{b}$ componentwise tight). Hence, if $\mathbf{1}^T\boldsymbol{\tau} < \beta$ the primal problem is infeasible. On the other hand, if $\mathbf{1}^T\boldsymbol{\tau} \geq \beta$ the primal problem is feasible, and a feasible point is given by (39). In order to show that this is indeed the desired solution, we shall assume that $\mathbf{1}^T\boldsymbol{\tau} \geq \beta$ and find a feasible point for the dual problem. Then, we show that the value of the dual problem at this point is equal to the value of the primal problem at (39).

We rewrite the inequality constraint and the objective function in the dual problem (45) as

$$\mathbf{A}^T \mathbf{y} \geq \alpha \mathbf{1} - \boldsymbol{\gamma} \quad (46)$$

and

$$-\mathbf{b}^T \mathbf{y} + \alpha \beta \quad (47)$$

The vector $\alpha \mathbf{1} - \boldsymbol{\gamma}$ has decreasing components. For fixed α , let K_α denote the number of positive elements of $\alpha \mathbf{1} - \boldsymbol{\gamma}$. It is clear from (46) and (47) that the objective function is maximized by letting the last $J - K_\alpha$ components of \mathbf{y} equal to zero. We introduce the following short-hand notation: for a vector $\mathbf{x} \in \mathbb{R}^J$ and a matrix $\mathbf{M} \in \mathbb{R}^{J \times J}$, we let \mathbf{x}_α and \mathbf{M}_α denote the $K_\alpha \times 1$ subvector of \mathbf{x} formed by its first K_α components, and the $K_\alpha \times K_\alpha$ submatrix of \mathbf{M} formed by its first K_α rows and columns, respectively. Then, a feasible point for the dual problem is the vector $\boldsymbol{\pi}$ such that its first K_α components are given by

$$\boldsymbol{\pi}_\alpha = \alpha (\mathbf{A}_\alpha^T)^{-1} \mathbf{1}_\alpha - (\mathbf{A}_\alpha^T)^{-1} \boldsymbol{\gamma}_\alpha \quad (48)$$

and the remaining $J - K_\alpha$ components are equal to zero.

The value of the objective function (47) at this point is given by

$$f(\alpha) = \mathbf{b}_\alpha^T (\mathbf{A}_\alpha^T)^{-1} \boldsymbol{\gamma}_\alpha + \alpha (\beta - \mathbf{b}_\alpha^T (\mathbf{A}_\alpha^T)^{-1} \mathbf{1}_\alpha) \quad (49)$$

It is not difficult to see that $f(\alpha)$ is a continuous and piecewise linear function of α , for $\alpha \leq [\gamma_1, \gamma_J]$. The assumption $\mathbf{1}^T\boldsymbol{\tau} \geq \beta$ can be rewritten as $\beta - \mathbf{1}^T\mathbf{A}^{-1}\mathbf{b} \leq 0$. Hence, for $\alpha > \gamma_s$, for some $1 \leq s \leq J$, the slope of $f(\alpha)$ is negative, while for $\alpha < \gamma_s$ the slope is positive. Therefore, the maximum of $f(\alpha)$ with respect to α is achieved for $\alpha = \gamma_s$ and, by definition, s is the minimum index in $1, \dots, J$ such that $\sum_{j=1}^s \tau_j \geq \beta$, i.e., $s = \hat{J}$ defined in (39). The primal objective function evaluated at the feasible point (39) is given by

$$(\gamma_1, \dots, \gamma_s, 0, \dots, 0) \mathbf{A}^{-1} \mathbf{b} + \gamma_s \left(\beta - \underbrace{(1, \dots, 1, 0, \dots, 0)}_s \mathbf{A}^{-1} \mathbf{b} \right)$$

It is immediate to see that this coincides with the dual objective function $f(\alpha)$ evaluated at $\alpha = \gamma_s$. Hence, we conclude that (39) is the sought solution.

Proof of Proposition 2. The proof follows immediately by observing that, for $\beta \leq b_J$, the program (42) can be reformulated as the \hat{J} -dimensional polymatroid program [23]

$$\begin{cases} \text{maximize} & \sum_{i=1}^{\hat{J}} \beta_i R_i \\ \text{subject to} & \sum_{i \in S} \beta_i \leq r(S), \quad \forall S \subseteq \{1, \dots, \hat{J}\} \\ & \boldsymbol{\beta} \geq \mathbf{0} \end{cases} \quad (50)$$

where the rank function $r(S)$ is defined by

$$r(S) = \sum_{i=1}^{\max\{S\}} \Delta_i$$

where $\Delta_i = b_i - b_{i-1}$ for $i = 1, \dots, \hat{J} - 1$ and $\Delta_{\hat{J}} = \beta - b_{\hat{J}-1}$. Since $(b_0, \dots, b_{\hat{J}-1}, \beta)$ is increasing, $r(S)$ is submodular.

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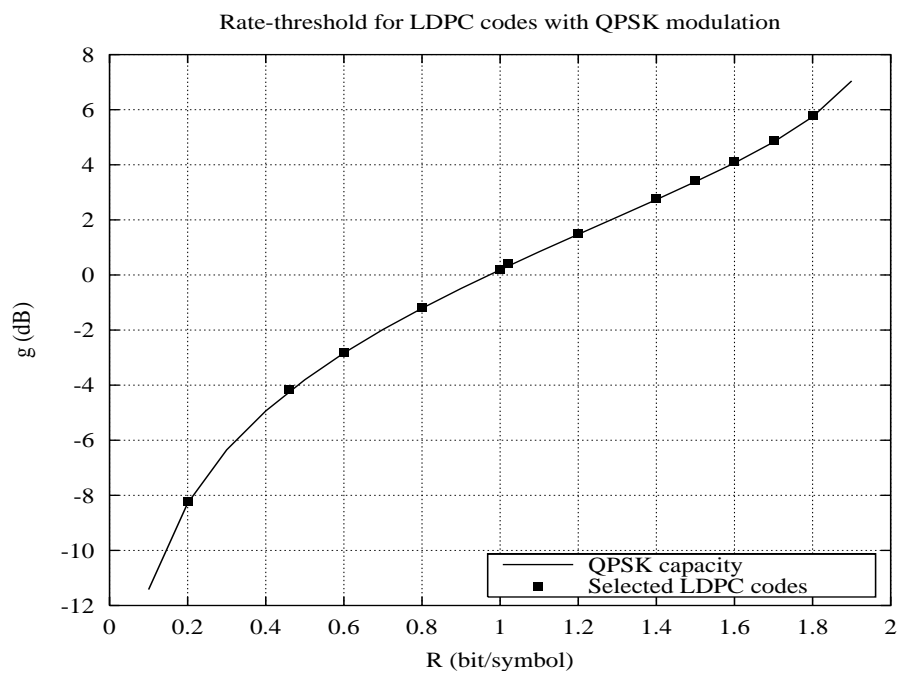


Figure 1: Rate-threshold pairs corresponding to QPSK capacity and for some LDPC codes from [7].

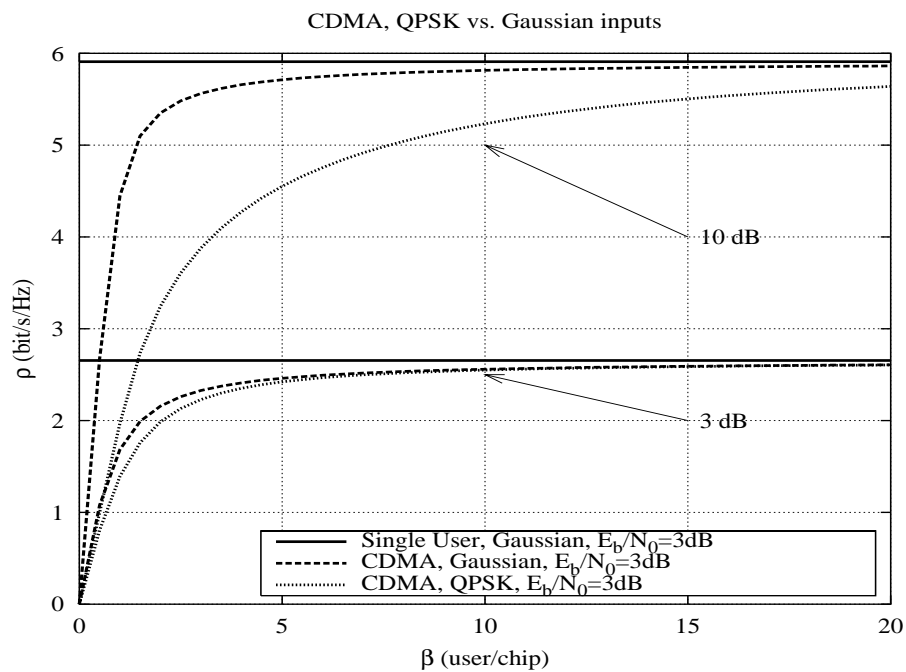


Figure 2: Spectral efficiency vs. β for random CDMA with Gaussian and QPSK inputs (with stripping decoder).

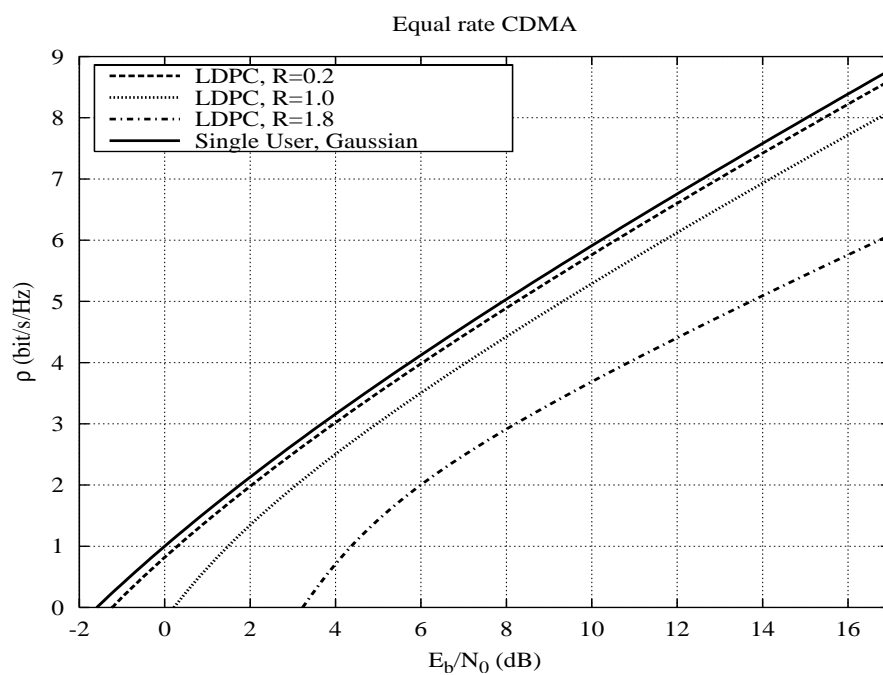


Figure 3: Spectral efficiency of some LDPC codes with equal-rate design.

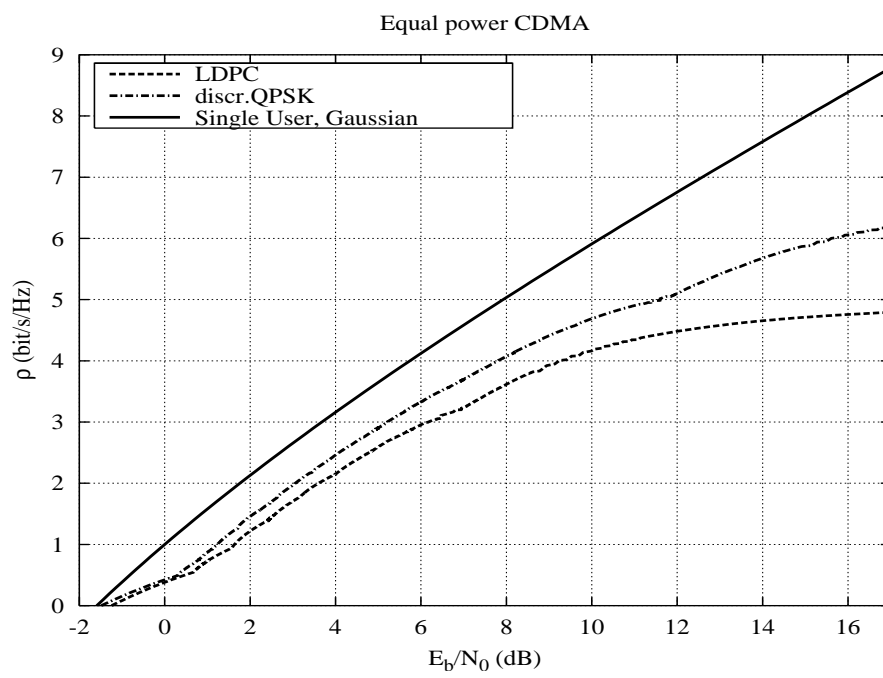


Figure 4: Spectral efficiency of LDPC and optimal QPSK codes with equal-power design.