ON THE EFFECT OF CHANNEL KNOWLEDGE IMPERFECTIONS AT THE TRANSMITTER ON THE CAPACITY OF MIMO SYSTEMS

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ABSTRACT

For a transmitter that has a perfect knowledge of the MIMO channel, the maximum achievable capacity corresponds to the water-filling solution. In practice, the available knowledge may only be partial due to the time selectivity of the channel, and delay or absence of the feedback from the receiver. However, exploiting the partial knowledge leads to a significant improvement when compared to the capacity without any channel knowledge. In this paper we analyze the MIMO capacity with various types of partial knowledge of the channel under practical frequency flat channel models.

1. INTRODUCTION

The introduction of Multi Input Multi Output (MIMO) systems leads to a significant increase in communication capacity. To take advantage of the use of MIMO systems, various space-time coding schemes have been proposed. These techniques assume the elements of the channel matrix to be i.i.d. In practice this assumption may not always be valid, since for physical reasons the channel components may be correlated [1]. This correlation corresponds to partial knowledge that can be fed back to the transmitter. When the partial channel knowledge is present at the transmitter, it is advantageous to use this information to optimize the precoder at the transmission [2, 3]. This precoder will basically be a cascade of space-time coder and a decorrelating beamformer.

In this paper, we investigate the achievable capacity given the available channel state information at the transmitter. We consider two different cases: MIMO pathwise channel model, and MIMO channel in the case of limited reciprocity. In the case of pathwise model we assume that the transmitter has information about slowly varying channel parameters, which may be used to calculate the channel correlations. In the case of limited reciprocity the transmitter knows the channel up to the amplitude and phase shifts that arise when the roles of transmitter and receiver are reversed. We demonstrate how the partial knowledge of the channel leads to an improvement of the communication capacity when compared to the capacity without any channel knowledge. However, the additional improvement when compared to knowing only the channel correlations is demonstrated to be small. We note that similar results (for different channel models) have also been published in [2].

Throughout this article scalar quantities are denoted by regular lowercase letters. Lower case bold type faces are used for vectors and regular uppercase letters for matrices. Superscripts T and H denote the transpose and conjugate transpose, respectively. We use diag$\{\boldsymbol{A}\}$ to denote the diagonal matrix of the diagonal elements of the matrix $\boldsymbol{A}$ and tr$\{\boldsymbol{A}\}$ (det$\{\boldsymbol{A}\}$) for the trace (determinant) of the matrix $\boldsymbol{A}$.

2. CHANNEL MODELS AND ASSUMPTIONS

We consider a MIMO communication system with N receive and M transmit antennas. The received $N \times 1$ signal vector is given by

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{v}, \tag{1}$$

where $\boldsymbol{H}$ is an $N \times M$ random channel matrix, $\boldsymbol{x}$ is an $M \times 1$ transmitted signal vector and $\boldsymbol{v}$ is an $N \times 1$ noise vector, which is assumed to be complex circular Gaussian with covariance matrix $\sigma_v^2 \boldsymbol{I}$. The channel covariance matrix at the transmitter is defined as $\boldsymbol{\Sigma} = E\{\boldsymbol{H}^H \boldsymbol{H}\}$. We use normalization tr$\{\boldsymbol{\Sigma}\} = 1$.

The ergodic capacity for the channel (1) is given by [5]

$$\mathcal{C} = E_H \left\{ \log \det \left[I + \rho \boldsymbol{H}\boldsymbol{Q}\boldsymbol{H}^H \right] \right\}, \tag{2}$$

where $\rho = \frac{P}{\sigma_v^2}$ is the SNR and $P\boldsymbol{Q}$ is the covariance matrix of the transmitted Gaussian signals maximizing the above expression, under the power constraint tr$\{\boldsymbol{Q}\} \leq 1$.

2.1. Pathwise channel model

The pathwise model [4] for the channel matrix in the case of frequency flat fading is

$$\boldsymbol{H} = \sum_{l=1}^{L} c_l \boldsymbol{a}_l \boldsymbol{b}_l^T, \tag{3}$$

where L is the number of multipaths and c_l, $i = 1, \ldots, L$ denote the complex multipath amplitudes. We assume that the amplitudes c_l are i.i.d. circular symmetric complex Gaussian distributed with mean 0 and variance 1. The $N \times 1$ vectors $\boldsymbol{a}_i$ are the steering vectors of the receive antenna array and the M-vectors $\boldsymbol{b}_l$ are the steering vectors of the transmitting antenna array. Due to the i.i.d.

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assumption of the complex amplitudes, it is assumed that the multipath variances are included in the vectors $\boldsymbol{b}_i$. We also normalize $||\boldsymbol{a}_i||^2 = 1\ \forall i$. Generally all c_l, $\boldsymbol{a}_l$ and $\boldsymbol{b}_l$ are random variables. The complex amplitudes c_l model the fast fading channel parameters and the steering vectors model the slowly fading channel parameters.

The channel matrix may also be given as

$$\boldsymbol{H} = \boldsymbol{ACB}, \tag{4}$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \dots, \boldsymbol{a}_L]$, $\boldsymbol{B} = [\boldsymbol{b}_1, \dots \boldsymbol{b}_L]^T$ and $\boldsymbol{C} = \text{diag}\{c_1, \dots, c_L\}$. If for every channel usage the receiver knows the realization of the channel and the slowly fading parameters remain constant over a sufficient time interval, the slowly fading parameters may be obtained at the receiver [6], and fed back to the transmitter. This information then corresponds to partial channel state information at the transmitter. We investigate the ergodic capacity of the channel given in (4) when $\boldsymbol{A}$ and $\boldsymbol{B}$ are fixed.

2.2. Channel models for limited reciprocity

Assume that the physical channel is reciprocal between uplink and downlink, and the transmitter knows the uplink channel $\boldsymbol{W}^T$. The overall channel in downlink including the cabling and electronic devices for both ends is therefore

$$\boldsymbol{H} = \boldsymbol{D}_1 \boldsymbol{W} \boldsymbol{D}_2,$$

where $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ are diagonal matrices. These matrices reflect the amplitude and phase shifts that arise when the roles of transmitter and receiver are reversed in case of no or limited calibration. We use three different models for the matrices $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$

Model 1 Only phase shifts: Diagonal elements contain i.i.d. phases ($\boldsymbol{D}_1 = \text{diag}\{e^{j\phi_1^1}, \dots, e^{j\phi_N^1}\}$ and $\boldsymbol{D}_2 = \text{diag}\{e^{j\phi_1^2}, \dots, e^{j\phi_M^2}\}$, where ϕ_l^i are i.i.d. and uniformly distributed on $[0, 2\pi]$)

Model 2 Case of complete absence of calibration: Diagonal elements of D_1 and D_2 are i.i.d. zero mean complex circularly symmetric Gaussian with variance 1.

Model 3 Case of imperfect calibration: The diagonal matrices are given by $\boldsymbol{D}_1 = \sqrt{1-\epsilon_1^2}\boldsymbol{I} + \epsilon_1 \boldsymbol{DN}_1$ and $\boldsymbol{D}_2 = \sqrt{1-\epsilon_2^2}\boldsymbol{I} + \epsilon_2 \boldsymbol{DN}_2$, where ϵ_i are small and $\boldsymbol{DN}_1$ and $\boldsymbol{DN}_2$ are diagonal matrices with i.i.d. diagonal elements that are zero mean complex circularly symmetric Gaussian with variance 1.

3. RESULTS FOR PATHWISE CHANNEL MODEL

In the case of pathwise model, the ergodic capacity for a given transmit covariance matrix $\boldsymbol{PQ}$ is

$$\mathcal{C} = E_C\left\{\log\det\left[I + \rho \boldsymbol{ACBQB}^H \boldsymbol{C}^H \boldsymbol{A}^H\right]\right\}. \tag{5}$$

For arbitrary SNR (ρ), the optimal $\boldsymbol{Q}$ can be given by direct numerical solution as described later in this paper. In what follows, we first give approximations for low and high SNR scenarios [7].

3.1. Low SNR

: for $\rho << 1$, the optimal transmit covariance matrix maximizing (5) is given by

$$\boldsymbol{Q} = \boldsymbol{u}\boldsymbol{u}^H, \tag{6}$$

where $\boldsymbol{u}$ is the eigenvector corresponding to the maximum eigenvalue of the channel covariance matrix

$$\boldsymbol{\Sigma} = \boldsymbol{B}^H \boldsymbol{B} = \boldsymbol{U\Lambda U}^H. \tag{7}$$

The optimal covariance matrix thus depends only on the channel covariance matrix at the transmitter.

3.2. High SNR

: for $\rho >> 1$, giving a general solution is not possible, because the optimal covariance matrix $\boldsymbol{Q}$ depends on the dimensions N, M and L, more specifically on the minimum dimension. The solution for two different possibilities for the minimum dimension is:

1. When $L \leq \min\{M, N\}$, the solution is given by

$$\boldsymbol{Q} = \frac{1}{L}\boldsymbol{U}\boldsymbol{U}^H, \tag{8}$$

where $\boldsymbol{U}$ is the matrix of the eigenvectors of $\boldsymbol{\Sigma}$ corresponding to the nonzero eigenvalues.

2. If $M \leq \min\{N, L\}$, the solution is given by

$$\boldsymbol{Q} = \frac{1}{M}\boldsymbol{I} \tag{9}$$

3.3. Waterfilling solution for the channel covariance matrix

Since log det is a concave on the set of positive definite matrices, the ergodic capacity for any transmit covariance matrix $\boldsymbol{Q}$ may be upper bounded by

$$\begin{aligned}\mathcal{C} &= E_C\left\{\log\det\left[I + \rho \boldsymbol{ACBQB}^H \boldsymbol{C}^H \boldsymbol{A}^H\right]\right\} \\ &\leq \log\det\left[I + \rho \boldsymbol{QB}^H E_C\{\boldsymbol{C}^H \boldsymbol{A}^H \boldsymbol{AC}\}\boldsymbol{B}\right] \\ &= \log\det\left[I + \rho \boldsymbol{QB}^H \boldsymbol{B}\right].\end{aligned}$$

The optimal $\boldsymbol{Q}$ maximizing this upper bound corresponds to the waterfilling solution applied to $\rho\boldsymbol{\Sigma}$ [5]. It can be shown that the waterfilling solution for $\rho << 1$ and $\rho >> 1$ matches the solutions given in equations (6),(8) and (9).

3.4. Optimal solution

As mentioned above, log det is concave on the set of positive definite matrices. The set of positive semidefinite matrices with trace equal to 1 is a convex set. Therefore, the optimum transmit covariance matrix may be found by using numerical methods. In practice, the object function has to be formed by averaging over sufficient number of Monte Carlo realizations. Note that the averaging preserves the concavity of the objective function. The applied method is based on projected gradient descent algorithm [8].

4. RESULTS FOR CHANNEL MODELS WITH LIMITED RECIPROCITY

In the case of limited reciprocity, the ergodic capacity for transmit covariance matrix $P\boldsymbol{Q}$ is

$$\mathcal{C} = E\left\{\log\det\left[I + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{Q}\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right]\right\}, \quad (10)$$

where the expectation is calculated with respect to $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$.

We first show that in the case of Model 1 or Model 2 (only phases or Gaussian zero mean diagonal entries), the optimal transmit covariance matrix has to be diagonal: $\boldsymbol{Q} = \boldsymbol{D}_Q$. Let $\boldsymbol{\Phi} = \text{diag}\{e^{j\phi_1}, e^{j\phi_2}, \ldots, e^{j\phi_M}\}$ with ϕ_i i.i.d. and uniformly distributed on $[0, 2\pi)$. Since for Models 1 and 2 the distribution of $\boldsymbol{D}_2$ is the same same as the distribution of $\boldsymbol{D}_2\boldsymbol{\Phi}$, the ergodic capacity may also be written as

$$E_\phi E_{D_1,D_2}\log\det\left[\boldsymbol{I} + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{\Phi}\boldsymbol{Q}\boldsymbol{\Phi}^H\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right].$$

Since log det is concave,

$$\begin{aligned}
&E_\phi E_{D_1,D_2}\log\det\left[\boldsymbol{I} + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{\Phi}\boldsymbol{Q}\boldsymbol{\Phi}^H\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right]\\
&\leq E_{D_1,D_2}\log\det\left[\boldsymbol{I} + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2 E_\phi\{\boldsymbol{\Phi}\boldsymbol{Q}\boldsymbol{\Phi}^H\}\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right]\\
&= E_{D_1,D_2}\log\det\left[\boldsymbol{I} + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2\text{diag}\{\boldsymbol{Q}\}\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right].
\end{aligned}$$

The equality is achieved if and only if $\boldsymbol{Q}$ is a diagonal matrix, and the result follows.

For the Model 1, the optimum solution may hence be derived by numerically maximizing

$$\mathcal{C} = \log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_Q\boldsymbol{W}^H\right], \quad (11)$$

which is a concave on $\boldsymbol{D}_Q$. We note that for given $\boldsymbol{D}_Q$, (11) is an upper bound of the ergodic capacity for Model 2.

For Model 2, the optimal solution can be found by using numerical methods described in Section 3.4, but the optimization is simpler because it has to be done only for diagonal matrices. For Model 3, optimization is performed as described in Section 3.4.

In addition to the optimal solutions, sub-optimal solutions may be derived by considering the upper bound on ergodic capacity as was done in the case of pathwise model in Section 3.3. For Models 1 and 2, this leads to waterfilling on

$$\rho\text{diag}\{\boldsymbol{W}^H\boldsymbol{W}\},$$

when for Model 3, it leads to waterfilling on

$$\rho\left((1-\epsilon_1^2)\boldsymbol{W}^H\boldsymbol{W} + \epsilon_1^2\text{diag}\{\boldsymbol{W}^H\boldsymbol{W}\}\right).$$

For Model 2, a tighter upper bound is given by (11). Therefore, a better solution may be given by applying the optimal solution for Model 1. For Model 3, waterfilling on $\rho\boldsymbol{W}^H\boldsymbol{W}$ can also be used.

4.1. Min-Max Problem

In the previous section we assume implicitly that the transmitter can see different realizations of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ (and therefore code on different realizations), without this assumption considering the ergodic capacity is meaningless. Below we assume that the transmitter can see only one realization of the channel, and depending on the way of encoding this leads to either a success or failure of the transmission. In a deterministic point of view (one realization, non-statistic problem), the transmitter should encode in such a manner as to be sure that the receiver will decode with success, this corresponds to an infinite failure cost. The problem then, can be solved in a deterministic manner by maximizing the worst case of the capacity over all possible values of $\boldsymbol{D}_1$, $\boldsymbol{D}_2$. The formulation of the Min-Max Problem is then:

$$\max_{\boldsymbol{Q}:tr\,\rho\boldsymbol{Q}\leq P}\ \min_{\boldsymbol{D}_1,\boldsymbol{D}_2}\log\det\left[I + \rho\boldsymbol{D}_1\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{Q}\boldsymbol{D}_2^H\boldsymbol{W}^H\boldsymbol{D}_1^H\right]. \quad (12)$$

As $\boldsymbol{0}$ is a possible value for $\boldsymbol{D}_1$, $\boldsymbol{D}_2$ in Models 2 and 3, it is therefore easy to see that the minimum capacity is zero for all values of $\boldsymbol{Q}$, and the Min-Max Problem formulation is not useful for Models 2 and 3. Below let's focus on the case of Model 1 (diagonals of phases).

We will derive Upper and Lower Bounds on (12) ($LB \leq$ (12) $\leq UB$), and shows that we obtain equality between the UB and LB (and therefore with (12)), which leads to the optimality of the LB (equivalently UB) solution.
The UB is obtained by:

$$\begin{aligned}
(12) &= \max_{\boldsymbol{Q}:tr\,\rho\boldsymbol{Q}\leq P}\ \min_{\boldsymbol{D}_2}\log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{Q}\boldsymbol{D}_2^H\boldsymbol{W}^H\right]\\
&\leq \max_{\boldsymbol{Q}:tr\,\rho\boldsymbol{Q}\leq P} E_{\boldsymbol{D}_2}\log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{Q}\boldsymbol{D}_2^H\boldsymbol{W}^H\right]\\
&\leq \max_{\boldsymbol{Q}:tr\,\rho diag\{\boldsymbol{Q}\}\leq P}\log\det\left[I + \rho\boldsymbol{W}diag\{\boldsymbol{Q}\}\boldsymbol{W}^H\right] = UB
\end{aligned}$$

the expectation $E_{\boldsymbol{D}_2}$, is done over the all possible values of $\boldsymbol{D}_2$, this give a larger capacity than $\min_{\boldsymbol{D}_2}\log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{Q}\boldsymbol{D}_2^H\boldsymbol{W}^H\right]$ and hence shows the first inequality. The second inequality follows from the concavity of $\log\det$.
The LB is obtained by maximizing over the subset of diagonals ($\boldsymbol{Q} = \boldsymbol{D}_Q$):

$$\begin{aligned}
(12) &\geq \max_{\boldsymbol{Q}:tr\,\rho\boldsymbol{D}_Q\leq P}\ \min_{\boldsymbol{D}_2}\log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_2\boldsymbol{D}_Q\boldsymbol{D}_2^H\boldsymbol{W}^H\right]\\
&\geq \max_{\boldsymbol{Q}:tr\,\rho\boldsymbol{D}_Q\leq P}\log\det\left[I + \rho\boldsymbol{W}\boldsymbol{D}_Q\boldsymbol{W}^H\right] = LB.
\end{aligned}$$

It is easy to see that the expressions of the upper and the lower bounds are the same. We conclude then that $LB = UB = (12)$, and that the solution of (12) matches the solution of LB, hence it is diagonal and corresponds to the same solution of the outage capacity optimization (11).

Models 2 and 3 can be adapted to the Min-Max Problem by modifying the distributions of $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ in a way that leads to a meaningful problem, for example by choosing a truncated Gaussian distribution that takes into account the most likely values or by choosing other bounded distributions.

5. SIMULATION RESULTS

Simulations were done in the case of limited reciprocity when using $N = M = 4$. The presented results are averaged over 100 realizations for $\boldsymbol{W}$, for which every element was generated independently from $\mathcal{CN}(0,1)$ distribution. For every realization of $\boldsymbol{W}$

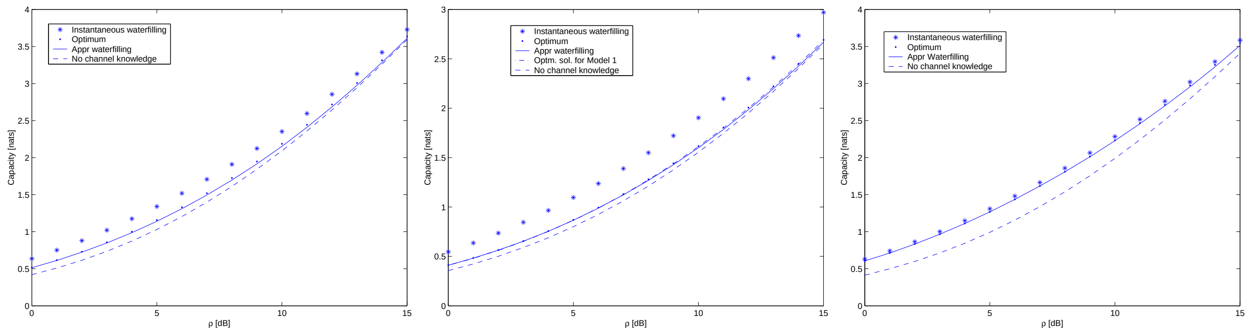

Figure 1: Results for limited reciprocity, $N = M = 4$. From left to right: Model 1, Model 2 and Model 3.

the capacities were averaged over 1000 Monte-Carlo realizations for $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$. For Model 3, we use $\epsilon_1^2 = \epsilon_2^2 = 0.1$.

Simulation results are presented in Figure 1. It can be seen that for Model 1 and Model 2, approximated waterfilling gives near optimal results. Therefore, waterfilling on the covariance matrix seen from the transmitter is almost sufficient. The same observation can be made also from the results for Model 3. Furthermore, the use of the solution of the Min-Max Problem for Model 1, achieves the optimal performance with Model 1 and very close to optimal for Model 2.

6. CONCLUSION

We studied the ergodic capacity of two models for partial channel knowledge: the pathwise channel model with knowledge of the slow varying parameters at the transmitter and the limited reciprocity channel model. The simulation studies and the theoretical results show that waterfilling on the channel covariance matrix at the transmitter leads to almost optimal capacity. We also introduced the Min-Max Problem for the limited reciprocity model, the solution of this problem in case of phases ambiguities leads to the same solution as the ergodic capacity maximization problem.

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