Euclidean Ordering via Chamfer Distance Calculations.

(Running head: "Euclidean ordering using chamfer distances")

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Abstract

This paper studies the mapping between continuous and discrete distances. The continuous distance considered is the widely used Euclidean distance whereas we consider as discrete distance the chamfer distance based on 3×3 masks.

A theoretical characterisation of topological errors which arise during the approximation of Euclidean distances by discrete ones is presented. Optimal chamfer distance coefficients are characterised with respect to the topological ordering they induce rather than the approximation of Euclidean distance values. We conclude the theoretical part by presenting a global upper bound for a topologically-correct distance mapping, irrespective of the chamfer distance coefficients, and identify the smallest coefficients associated with this bound.

We use these results to solve a problem which is a representative of most of problems in image processing, namely the Euclidean-nearest neighbour problem. This problem is formulated as a discrete optimisation problem and solved accordingly using algorithmic graph theory and integer arithmetic.

1 Introduction

The main motivation of this work is to analyse the mapping between continuous and discrete distances on the unit square grid. For a given pixel on the grid, a neighbourhood of that pixel is defined by a distance metric. The local neighbours of a pixel are contained within a mask centred at that pixel. The most commonly used is the 3×3 mask, where the neighbours are typically defined by the City-Block distance (4-neighbours) or the Chessboard distance (8-neighbours) [11]. Local distances within the mask form the basis for the computation of global distances on the grid. Chamfer distance and was first introduced by Hilditch [7, 8] and studied by Borgefors in [1] and [2] for the approximation of Euclidean distances on the grid.

While studying in depth the calculation of Euclidean distance values using discrete distance functions (Section 2), we will derive results concerning the decomposition of integer values which can then form the basis for the development of optimal algorithms for the solution of typical problems encountered in image processing (see *e.g.*, [9]). In Section 3, we propose to apply theoretical results derived in this paper and tools issued from graph theory to solve exactly and optimally the Euclidean-nearest neighbour problem (*i.e.*, where continuous distances values are required) using integer arithmetic. Solutions of the nearest neighbour problem have applications for the computation of different structures in image processing (*e.g.*, distance maps and Voronoi diagrams). The idea behind the proposed solution therefore defines a new context in which these problems can be solved.

2 Topological Errors

We consider throughout these pages that the continuous distance used is the Euclidean distance d_{E} defined as $d_{\mathrm{E}}(p,q) = \sqrt{(x_p \Leftrightarrow x_q)^2 + (y_p \Leftrightarrow y_q)^2}$, where $p = (x_p, y_p)$ and $q = (x_q, y_q)$. We introduce the notation for some standard functions. $\lceil x \rceil$ is the smallest integer greater or equal to than $x \in \mathbb{R}$ and $\lfloor x \rfloor$ is greatest integer smaller or equal to than $x \in \mathbb{R}$. Then, $\mathrm{round}(x) = \lfloor x \rfloor$ if $\lfloor x \rfloor \leq x \leq \lfloor x \rfloor + \frac{1}{2}$. Otherwise, $\mathrm{round}(x) = \lceil x \rceil$.

In approximating the Euclidean distance, the chamfer distance is used since it allows for algorithms to operate in integer arithmetic. We will consider chamfer distances in relation to 3×3 masks. We define a as the length of the unit horizontal/vertical move (a-move) on the grid, and b as the length of the unit diagonal move (b-move) on the grid. Parameters a and b are referred to as distance transform (DT) coefficients. The chamfer distance can be computed as the length of a shortest path on the grid. Given two points p and q, the chamfer distance $d_{a,b}$ between p and q is calculated as follows.

$$d_{a,b}(p,q) = k_{\mathrm{a}} a + k_{\mathrm{b}} b \tag{1}$$

where k_{a} and k_{b} represent the number of a- and b-moves on the shortest path from p to q on the grid. The conditions on a and b for $d_{a,b}$ to be a distance are given in (2) below (see [10] and [14] for more details). Such conditions allow $d_{a,b}$ to satisfy the three common metric properties.

$$0 < a < b < 2a \tag{2}$$

The number of a- and b-moves $(k_{\mathrm{a}}, k_{\mathrm{b}})$ on the shortest path from p to q can also be used to compute the Euclidean distance between p and q.

$$d_{\mathrm{E}}(p,q) = \sqrt{(k_{\mathrm{a}} + k_{\mathrm{b}})^2 + k_{\mathrm{b}}^2} \tag{3}$$

Without loss of generality, we restrict this study to values of DT coefficients such that a and b are relatively prime (*i.e.*, the Greater Common Divisor of a and b, $\gcd(a,b)$, is such that $\gcd(a,b) = 1$). This corresponds to normalising the a and b coefficients to their minimal configuration.

Although different studies establish particular values of DT coefficients as optimal (*e.g.*, $a = 3$, $b = 4$), optimality generally refers to a criterion in relation to a minimum error in the approximation of Euclidean *distance values* via chamfer distance computation (see *e.g.*, [2, 14, 15]). In contrast, Forchhammer [5, 6] introduced the concept of *topological inconsistencies* induced by discrete distances when used as an approximation of the Euclidean distance. The inconsistency arises because of the difference in ordering of the two distance measures as will be described next. He derived values of DT coefficients a and b which are empirically shown to be optimal with regard to this criterion. The results were based on the use of look-up tables and an investigation of a limited number of values of the parameters under study.

In this section, we propose an analytical (and therefore, complete) characterisation of the optimal values of a and b with respect to the *topological* criterion. Essentially, the ordering of the discrete distance does not match the ordering of the Euclidean distance. Consider the following example (see Figure 1). Let the DT coefficients be $a = 2$, $b = 3$, and consider the three integer points (pixels), $p = (0,0)$, $q = (10,1)$ and $r = (9,4)$. The shortest path on the grid from p to q is given by $k_{\mathrm{a}} = 9$ and $k_{\mathrm{b}} = 1$ and that from p to r is given by $k_{\mathrm{a}} = 5$ and $k_{\mathrm{b}} = 4$. Using Equations (1) and (3), we have $d_{a,b}(p,q) = 21$, $d_{\mathrm{E}}(p,q) = \sqrt{101}$, $d_{a,b}(p,r) = 22$ and $d_{\mathrm{E}}(p,r) = \sqrt{97}$. If q and r are border pixels, the discrete DT will lead to consider q as the nearest border pixel to p (by the chamfer distance measure) giving an approximate Euclidean distance of $\sqrt{101}$. This is clearly incorrect since there is a smaller Euclidean distance between p and another border pixel (namely r) giving a Euclidean distance of $\sqrt{97}$. In other words, since, $d_{a,b}(p,q) < d_{a,b}(p,r)$ and $d_{\mathrm{E}}(p,q) > d_{\mathrm{E}}(p,r)$, the ordering of $d_{a,b}$ differs from the ordering of d_{E}.

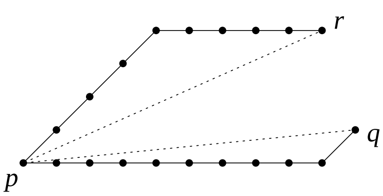

Figure 1: An example of a topological error.

Given a pair of DT coefficients (a,b), we characterise the configurations for which this problem occurs precisely. First, we introduce how restrictions for the decomposition of a given discrete distance value D into a- and b-moves can be given by the solution to the Frobenius problem.

Theorem 1 *[13]. Given* $0 < a < b$ *such that* $(a,b) \in \mathbb{N}^2$ *and* $\gcd(a,b) = 1$. *Consider the equation:* $k_{\mathrm{a}}a + k_{\mathrm{b}}b = D \quad (k_{\mathrm{a}}, k_{\mathrm{b}}) \in \mathbb{N}^2$ *If* $\chi = (a \Leftrightarrow 1)(b \Leftrightarrow 1)$, *then we have the following instances.*

(i) If $D \geq \chi$, *there is always at least one solution* $(k_{\mathrm{a}}, k_{\mathrm{b}})$.

(ii) If $D = \chi \Leftrightarrow 1$, *there is no solution.*

(iii) There is exactly $\frac{1}{2}\chi$ *values of* D *that have no solution.*

We will use the solution to this classical problem to characterise the topological error introduced earlier. Two types of errors are distinguished and presented in Sections 2.1 and 2.2, respectively.

2.1 Type 1 error

Given a pair of DT coefficients (a, b) and three integer points p, q and r, a Type 1 error occurs between q and r relative to p if $d_{a,b}(p, q) = d_{a,b}(p, r)$ and $d_{\mathrm{E}}(p, q) \neq d_{\mathrm{E}}(p, r)$. More formally, we make the following definition.

Definition 1 *Type 1 error. Given a pair of DT coefficients (a, b) and a discrete distance value D, a Type 1 topological error occurs if there exist two integer pairs $(k_{\mathrm{a}_1}, k_{\mathrm{b}_1})$ and $(k_{\mathrm{a}_2}, k_{\mathrm{b}_2})$ such that $k_{\mathrm{a}_1} a + k_{\mathrm{b}_1} b = k_{\mathrm{a}_2} a + k_{\mathrm{b}_2} b = D$ and $\sqrt{(k_{\mathrm{a}_1} + k_{\mathrm{b}_1})^2 + k_{\mathrm{b}_1}^2} \neq \sqrt{(k_{\mathrm{a}_2} + k_{\mathrm{b}_2})^2 + k_{\mathrm{b}_2}^2}$.*

An example for Type 1 error is illustrated in Figure 2 where, $(k_{\mathrm{a}_1}, k_{\mathrm{b}_1})$ represents the shortest path from p to q, and $(k_{\mathrm{a}_2}, k_{\mathrm{b}_2})$ the shortest path from p to r. In this example, the DT coefficients are $a = 2$ and $b = 3$, and the three integer points are $p = (0, 0)$, $q = (3, 0)$ and $r = (2, 2)$. We obtain $d_{a,b}(p, q) = d_{a,b}(p, r) = D = 6$, since $k_{\mathrm{a}_1} = 3$, $k_{\mathrm{b}_1} = 0$ and $k_{\mathrm{a}_2} = 3$, $k_{\mathrm{b}_2} = 0$. On the other hand, we have, $d_{\mathrm{E}}(p, q) = 3$ and $d_{\mathrm{E}}(p, r) = \sqrt{8}$.

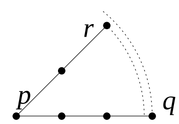

Figure 2: The first instance of Type 1 topological error for $(a, b) = (2, 3)$.

It is clear that a necessary condition for Type 1 error to occur is that D can be decomposed in more than one manner. Lemma 1 gives an upper bound on the number of such decompositions, and Lemma 2 provides an exact upper bound on the number of b-moves ($k_{\mathrm{b}_{\max}}$) over all possible decompositions.

Lemma 1 *Given a pair of DT coefficients (a, b) and a discrete distance value $D \in \mathbb{N}$, there is at most $\left\lfloor \frac{\lfloor \frac{D}{b} \rfloor + 1}{a} \right\rfloor + 1$ configurations $(k_{\mathrm{a}}, k_{\mathrm{b}})$ such that $k_{\mathrm{a}} a + k_{\mathrm{b}} b = D$ with $k_{\mathrm{a}} \geq 0$ and $k_{\mathrm{b}} \geq 0$.*

Proof:
Clearly, k_{b} can take at most $\left\lfloor \frac{D}{b} \right\rfloor + 1$ values. Now, among these values, only some can satisfy the equation $k_{\mathrm{a}} a + k_{\mathrm{b}} b = D$. Let us assume that $(k_{\mathrm{a}}, k_{\mathrm{b}})$ is one such valid pair. Then, $D = k_{\mathrm{a}} a + k_{\mathrm{b}} b = k_{\mathrm{a}} a + k_{\mathrm{b}} b + ab \Leftrightarrow ab = (k_{\mathrm{a}} + b)a + (k_{\mathrm{b}} \Leftrightarrow a)b$. Then, if $k_{\mathrm{b}} \Leftrightarrow a \geq 0$, the configuration $(k_{\mathrm{a}} + b, k_{\mathrm{b}} \Leftrightarrow a)$ is also valid. Likewise, all configurations $(k_{\mathrm{a}} + ib, k_{\mathrm{b}} \Leftrightarrow ia)$ with $i \in \mathbb{Z}$ such that $k_{\mathrm{a}} + ib \geq 0$ and $k_{\mathrm{b}} \Leftrightarrow ia \geq 0$ are valid. Clearly, since $\gcd(a, b) = 1$, there is at most $\left\lfloor \frac{\lfloor \frac{D}{b} \rfloor + 1}{a} \right\rfloor + 1$ such i values. Therefore, Lemma 1 holds. $\Box$

Lemma 2 *Given a pair of DT coefficients (a, b) and a discrete distance value D, we assume that the existence condition (i) in Theorem 1 holds. Then, the maximal value $k_{\mathrm{b}_{\max}}$ of k_{b} such that $k_{\mathrm{a}} a + k_{\mathrm{b}_{\max}} b = D$ with $k_{\mathrm{a}} \geq 0$ and $k_{\mathrm{b}_{\max}} \geq 0$ is given by:*

$$k_{\mathrm{b}_{\max}} = \left\lfloor \frac{D}{b} \right\rfloor \Leftrightarrow \psi((D \bmod b) \bmod a) \tag{4}$$

where ψ is the implicit integer function such that:
$\psi : \{0, 1, \cdots, a \Leftrightarrow 1\} \mapsto \{0, 1, \cdots, a \Leftrightarrow 1\}$ *and* $\psi((x.(2a \Leftrightarrow b)) \bmod a) = x$.

Proof:
The existence of a decomposition is given by Theorem 1. The problem of finding the maximal value $k_{\mathrm{b}_{\max}}$ of k_{b} such that $k_{\mathrm{a}}a + k_{\mathrm{b}_{\max}}b = D$ can be mapped onto the problem of finding the minimal positive value k^* of k such that

$$(D \bmod b + kb) \bmod a = 0 \tag{5}$$

In this case, we have:

$$k_{\mathrm{b}_{\max}} = \left\lfloor \frac{D}{b} \right\rfloor \Leftrightarrow k^* \tag{6}$$

Consider the following implicit integer function $\psi : \mathbb{N} \rightarrow \mathbb{N}$ in its general form.

$$\psi(x) = \begin{cases} k & \text{if} \quad (x+kb) \bmod a = 0 \\ +\infty & \text{if} \quad \forall k \geq 0 \;\; (x+kb) \bmod a \neq 0 \end{cases} \tag{7}$$

Using the above definition and modulus arithmetic, one can easily prove the following properties for ψ: *(i)* $\psi(0) = 0$, *(ii)* $\psi(x+a) = \psi(x)$, *(iii)* $\psi(x+b) = \psi(x) \Leftrightarrow 1$, *(iv)* $\psi(x + (2a \Leftrightarrow b)) = \psi(x) + 1$, *(v)* $\psi(x) = k \Leftrightarrow \psi(x) = k \bmod a$ Hence, from Properties *(i)* and *(ii)*, we deduce that ψ is periodic and, therefore, can be defined only for $x \in \{0, 1, \cdots, a \Leftrightarrow 1\}$.

Now, let $k \in \mathbb{N}$ and $k' \in \mathbb{N}$ be such that $k \neq k'$ and $(kb) \bmod a = (k'b) \bmod a = \alpha$. Then, $\exists \beta,\ \gamma \in \mathbb{N}$ such that $kb = \alpha + \beta a$ and $k'b = \alpha + \gamma a$. It follows that,

$$\begin{aligned} (kb) \bmod a = (k'b) \bmod a \quad & \Leftrightarrow \quad (k \Leftrightarrow k')b = (\beta \Leftrightarrow \gamma)a \\ & \Leftrightarrow \quad (k \Leftrightarrow k')b \bmod a = 0 \\ & \Leftrightarrow \quad (k \Leftrightarrow k') \bmod a = 0 \text{ since } \gcd(a,b) = 1 \end{aligned}$$

Equivalently, if $(k \Leftrightarrow k') \bmod a \neq 0 \Leftrightarrow (kb) \bmod a \neq (k'b) \bmod a$. Therefore, the set $\{(kb) \bmod a,\ \forall k \geq 0\}$ contains exactly a different values.

From Property *(v)*, we know that $\psi(x) \in \{0, 1, \cdots, a \Leftrightarrow 1\}$. Therefore, we can claim that, $\forall k \in \{0, 1, \cdots, a \Leftrightarrow 1\}$, $\exists x \in \{0, 1, \cdots, a \Leftrightarrow 1\}$ such that $\psi(x) = k$.

Therefore, ψ is a bijection from $\{0, 1, \cdots, a \Leftrightarrow 1\}$ to $\{0, 1, \cdots, a \Leftrightarrow 1\}$ such that $\psi((k(2a \Leftrightarrow b)) \bmod a) = k$. Hence, using Equations (5) and (7), we obtain that $k^* = \psi((D \bmod b) \bmod a)$. Therefore, combining this result with Equation (6), Lemma 2 holds. □

ψ can be easily computed as a one-to-one mapping of the set $\{0, 1, \cdots, a \Leftrightarrow 1\}$ onto itself. Thus, Lemma 2 readily gives an exhaustive list of $(k_{\mathrm{a}}, k_{\mathrm{b}})$ pairs for decomposing any discrete distance value D for any DT coefficients a and b.

Definition 2 *Given a pair of DT coefficients (a, b) and a discrete distance value D which can be decomposed in at least one manner, we define:*

- *The set $\Theta = \{(k_{\mathrm{a_i}}, k_{\mathrm{b_i}}), i = 0, \cdots, n\}$ as the exhaustive list of all possible decomposition pairs (i.e., $D = k_{\mathrm{a_i}}a + k_{\mathrm{b_i}}b$, $k_{\mathrm{a_i}} \geq 0$, $k_{\mathrm{b_i}} \geq 0$ $\forall i = 0, \cdots, n$ with $n = \left\lfloor \frac{k_{\mathrm{b}_{\max}}}{a} \right\rfloor$). Note that $k_{\mathrm{b}_{\max}} = \max_{i=0,\cdots,n} k_{\mathrm{b_i}}$ and, $k_{\mathrm{b_i}} = k_{\mathrm{b}_{\max}} \Leftrightarrow ia$. Therefore, the set Θ can be fully computed using Lemma 2.*

- *$R_i(D)$ as the Euclidean distance associated with the pair $(k_{\mathrm{a_i}}, k_{\mathrm{b_i}})$. From Equation (3), we have,*

$$R_i(D) = \sqrt{(k_{\mathrm{a_i}} + k_{\mathrm{b_i}})^2 + k_{\mathrm{b_i}}^2} \tag{8}$$

- $R_{\max}(D)$ *(resp.* $R_{\min}(D)$*) as the maximal (resp. minimal) Euclidean distance over all* $n+1$ *possible decompositions.*

- l *(resp.* m*) as the index in the set* Θ *of the decomposition* $(k_{\mathrm{a_l}}, k_{\mathrm{b_l}})$ *(resp.* $(k_{\mathrm{a_m}}, k_{\mathrm{b_m}})$*) leading to* $R_{\max}(D)$ *(resp.* $R_{\min}(D)$*).*

We now present a lemma which allows us to calculate the indices m and l (and, therefore, $R_{\max}(D)$ and $R_{\min}(D)$) directly for any DT coefficients (a,b) and any discrete distance value D.

Lemma 3 *Given a pair of DT coefficients* (a,b) *and a discrete distance value* D*, we assume that* D *can be decomposed in at least more than one manner.*

Then, the minimal decomposition (k_{a_m}, k_{b_m}) *leading to* $R_{\min}(D)$ *is given by*

$$m = \mathrm{round}\left(\frac{D(a \Leftrightarrow b)}{a(b^2 \Leftrightarrow 2.ab + 2a^2)} + \frac{k_{\mathrm{b_{max}}}}{a}\right) \tag{9}$$

and the maximal decomposition (k_{a_l}, k_{b_l}) *leading to* $R_{\max}(D)$ *is given by*

$$l = \begin{cases} 0 & \textit{if} \quad k_{\mathrm{b_{max}}} + (k_{\mathrm{b_{max}}} \bmod a) > \frac{2.D(b-a)}{b^2-2.ab+2a^2} \\ n & \textit{if} \quad k_{\mathrm{b_{max}}} + (k_{\mathrm{b_{max}}} \bmod a) \leq \frac{2.D(b-a)}{b^2-2.ab+2a^2} \end{cases} \tag{10}$$

Proof:
Re-arranging $D = k_{a_i}a + k_{b_i}b$, we obtain $k_{a_i} = \frac{D-k_{b_i}b}{a}$. Combining this with Equation (8), we obtain

$$R_i^2(D) = \frac{1}{a^2}\left[k_{b_i}^2(b^2 \Leftrightarrow 2.ab + 2.a^2) + 2.k_{b_i}D(a \Leftrightarrow b) + D^2\right] \tag{11}$$

Given i and j in $\{0, \cdots, n\}$ such that $i < j$, we have $k_{b_i} > k_{b_j}$. Then, $R_i(D) > R_j(D)$ if

$$k_{b_i} + k_{b_j} > \frac{2.D(b \Leftrightarrow a)}{b^2 \Leftrightarrow 2.ab + 2.a^2} \tag{12}$$

Now, if $k_{b_i} > k_{b_j}$ $\forall i < j$, Equation (12) shows that we have the following pattern:

$$R_0(D) > R_1(D) > \cdots > R_m(D) (= R_{\min}(D)) < R_{m+1}(D) < \cdots < R_n(D) \tag{13}$$

Therefore, m is the first integer such that $R_{m-1}(D) > R_m(D)$ and $R_m(D) \leq R_{m+1}(D)$:

$$k_{\mathrm{b_{max}}} \Leftrightarrow (m \Leftrightarrow 1).a + k_{\mathrm{b_{max}}} \Leftrightarrow m.a > \frac{2.D(b \Leftrightarrow a)}{b^2 \Leftrightarrow 2.ab + 2.a^2}$$

and

$$k_{\mathrm{b_{max}}} \Leftrightarrow m.a + k_{\mathrm{b_{max}}} \Leftrightarrow (m+1)a \leq \frac{2.D(b \Leftrightarrow a)}{b^2 \Leftrightarrow 2.ab + 2.a^2}$$

Therefore,

$$\frac{D(a \Leftrightarrow b)}{a(b^2 \Leftrightarrow 2.ab + 2a^2)} + \frac{k_{\mathrm{b_{max}}}}{a} + \frac{1}{2} > m \geq \frac{D(a \Leftrightarrow b)}{a(b^2 \Leftrightarrow 2.ab + 2a^2)} + \frac{k_{\mathrm{b_{max}}}}{a} \Leftrightarrow \frac{1}{2}.$$

Hence, Equation (9) holds. It is important to note that m is uniquely defined by the above equation. Using the result in Equation (13), we can also say that the maximum Euclidean distance is obtained for either extreme of i. Using Lemma 1, we obtain that $k_{b_0} = k_{\mathrm{b_{max}}}$ and $k_{b_n} = k_{\mathrm{b_{max}}} \bmod a$. Therefore, using the valid inequality given in Equation (12), we can claim that Equation (10) completely characterises l. Hence, Lemma 3 holds. □

Lemma 4 *Characterisation of a Type 1 error. Given a pair of DT coefficients (a, b), a Type 1 error occurs for any discrete distance value D for which $R_{\max}(D) \neq R_{\min}(D)$.*

Clearly, the first instance of D for which $R_{\max}(D) \neq R_{\min}(D)$ is $D = ab$. In this case, $k_{\mathrm{b}_{\max}} = a$, $n = 1$, $\Theta = \{(0, a), (b, 0)\}$, (*i.e.*, $R_0 = a\sqrt{2}$ and $R_1 = b$). Hence, $R_{\max}(ab) \neq R_{\min}(ab)$. Therefore, we define the following Euclidean distance limit when considering Type 1 errors only.

Definition 3 *Euclidean distance limit induced by Type 1 error, $\mathcal{R}_1(a, b)$. Given a pair of DT coefficients (a, b), and $D = ab$ as the minimum discrete distance value for which $R_{\min}(D) \neq R_{\max}(D)$, we define the Euclidean distance limit $\mathcal{R}_1(a, b)$ for Type 1 errors as follows. $\mathcal{R}_1(a, b)$ is the maximal Euclidean distance value deduced from a discrete distance value (i.e., using Equation (3)) up to which both discrete and continuous distance ordering match. More formally, $\mathcal{R}_1(a, b)$ is the maximal Euclidean distance value R such that $\exists\ k_{\mathrm{a}},\ k_{\mathrm{b}} \in \mathbb{N}$ such that $R = \sqrt{(k_{\mathrm{a}} + k_{\mathrm{b}})^2 + k_{\mathrm{b}}^2}$ and $R < R_{\max}(ab)$.*

In other words, $R_{\max}(ab)$ can be considered as a strict (*i.e.*, non-feasible) Euclidean distance limit. In order to obtain a feasible limit, we search D' the maximal discrete distance value such that $R_{\min}(D') < R_{\max}(ab)$ and consider $\mathcal{R}_1(a, b) = R_{\min}(D')$. Using the previous study, we can easily design an algorithm to compute, for any pair of DT coefficients (a, b), the value of $\mathcal{R}_1(a, b)$. In Figure 3, $(\mathcal{R}_1(a, b))^2$ is plotted for each pair of valid DT coefficients such that $a \leq 10$.

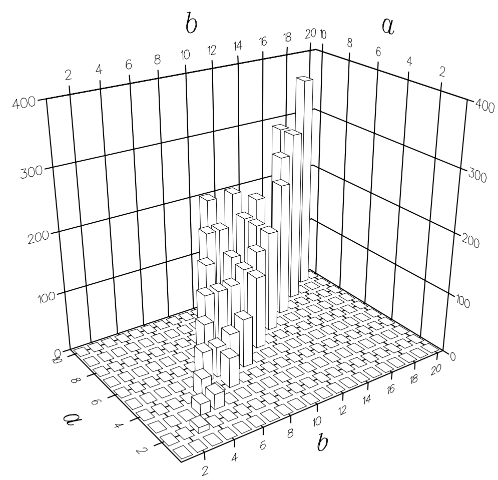

Figure 3: Euclidean distance limit induced by Type 1 of topological error $(\mathcal{R}_1(a, b))^2$.

2.2 Type 2 error

Given a pair of DT coefficients (a, b) and three integer points p, q and r, a Type 2 error occurs between q and r, relative to p, if $d_{a,b}(p, q) < d_{a,b}(p, r)$ and $d_{\mathrm{E}}(p, q) > d_{\mathrm{E}}(p, r)$.

Definition 4 *Type 2 error. Given a pair of DT coefficients (a, b) and two discrete distance values D_1 and D_2, we assume that $\exists\ (k_{\mathrm{a}_1}, k_{\mathrm{b}_1})$ and $(k_{\mathrm{a}_2}, k_{\mathrm{b}_2})$ such that $k_{\mathrm{a}_1} a + k_{\mathrm{b}_1} b = D_1$ and $k_{\mathrm{a}_2} a + k_{\mathrm{b}_2} b = D_2$ (see Theorem 1). A Type 2 error occurs if, $D_1 < D_2$ and $\sqrt{(k_{\mathrm{a}_1} + k_{\mathrm{b}_1})^2 + k_{\mathrm{b}_1}^2} > \sqrt{(k_{\mathrm{a}_2} + k_{\mathrm{b}_2})^2 + k_{\mathrm{b}_2}^2}$.*

Figure 4 illustrates an instance of Type 2 error between q and r, relative to p. In this example, the DT coefficients are $a = 2$ and $b = 3$. We consider $p = (0, 0)$, $q = (6, 0)$ and $r = (5, 3)$. Therefore, $D_1 = d_{a,b}(p, q) = 12$, since $k_{\mathrm{a}_1} = 6$ and $k_{\mathrm{b}_1} = 0$. We also obtain $D_2 = d_{a,b}(p, r) = 13$, since $k_{\mathrm{a}_2} = 2$ and $k_{\mathrm{b}_2} = 3$. On the other hand, we have $d_{\mathrm{E}}(p, q) = \sqrt{36}$ and $d_{\mathrm{E}}(p, r) = \sqrt{34}$. Therefore, $d_{a,b}(p, q) < d_{a,b}(p, r)$ and $d_{\mathrm{E}}(p, q) > d_{\mathrm{E}}(p, r)$.

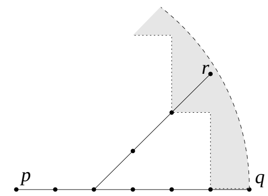

Figure 4: The first instance of Type 2 topological error for $(a, b) = (2, 3)$.

From a geometric viewpoint, given three integer points p, q and r such that $D_1 = d_{a,b}(p, q) < D_2 = d_{a,b}(p, r)$, a Type 2 error occurs between q and r, relative to p, if there exists at least one integer point (namely, r) included in the area between the discrete disc of radius D_1 centred at p and the Euclidean disc that contains this discrete disc. In Figure 4, this area is illustrated by the shaded surface. A Type 2 error occurs between q and r, relative to p, since r lies in this shaded surface.

The geometrical characterisation of Type 2 errors will, therefore, be investigated through the characterisation of the radius of the smallest Euclidean disc that contains the discrete disc of radius D for any given values of the DT coefficients (a, b) and for any discrete distance value D. The radius of such a Euclidean disc was noted $R_{\mathrm{max}}(D)$ (see Definition 2). Therefore, the geometrical characterisation of Type 2 error can be formally written as follows.

Lemma 5 *Given a pair of DT coefficients (a, b) and a discrete distance value D, a Type 2 error occurs in the discrete disc of radius D if there exists a discrete distance value $D' > D$ such that $R_{\mathrm{min}}(D') < R_{\mathrm{max}}(D)$.*

Using this characterisation, the Euclidean distance limit $\mathcal{R}_2(a, b)$ induced by Type 2 error can be defined as follows.

Definition 5 *Euclidean distance limit induced by Type 2 error, $\mathcal{R}_2(a, b)$ Given a pair of DT coefficients (a, b), and D, the minimum discrete distance value for which there exists a discrete distance value D' such that $R_{\mathrm{min}}(D') < R_{\mathrm{max}}(D)$, we define the Euclidean distance limit $\mathcal{R}_2(a, b)$ for Type 2 errors as follows.*

$\mathcal{R}_2(a, b) = R_{\mathrm{min}}(D')$ where D' is the smallest discrete distance value such that $R_{\mathrm{min}}(D') < R_{\mathrm{max}}(D)$.

In other words, if a Type 2 error occurs for the discrete distance value D (*e.g.*, at point q in Figure 4, with $D = 12$), we consider the Euclidean distance limit as the value $R_{\mathrm{min}}(D')$ where D' is the discrete distance value at the second point for which Type 2 error occurred (*e.g.*, point r in Figure 4, and $D' = 13$). In Figure 5, $(\mathcal{R}_2(a, b))^2$ is plotted for each DT coefficients pair such that $a \leq 10$.

Using the results of Sections 2.1 and 2.2, we can now define a combined Euclidean distance limit where no topological error of any type can occur.

Definition 6 *Global Euclidean distance limit, $\mathcal{R}(a, b)$. Given a pair of DT coefficients (a, b), we define the global Euclidean distance limit $\mathcal{R}(a, b)$ as the minimum between the distance limits induced by both Type 1 and 2 errors. Therefore, $\mathcal{R}(a, b) = \min(\mathcal{R}_1(a, b), \mathcal{R}_2(a, b))$.*

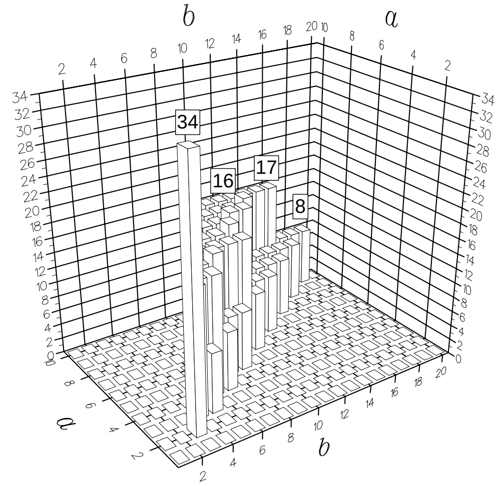

Figure 5: Distance limit induced by Type 2 error.

$\mathcal{R}(a, b)$ represents the maximal achievable Euclidean distance when growing topologically correct discrete discs. Equivalently, given a pair of DT coefficients (a, b), for any discrete distance value D such that $R_{\max}(D) \leq \mathcal{R}(a, b)$, no topological error (of Type 1 or Type 2) occurs in the discrete disc of radius D. In Figure 6, $(\mathcal{R}(a, b))^2$ is plotted for each DT coefficients pair such that $a \leq 10$. Note that, for large values of a and b, the limit induced by the Type 2 error dominates.

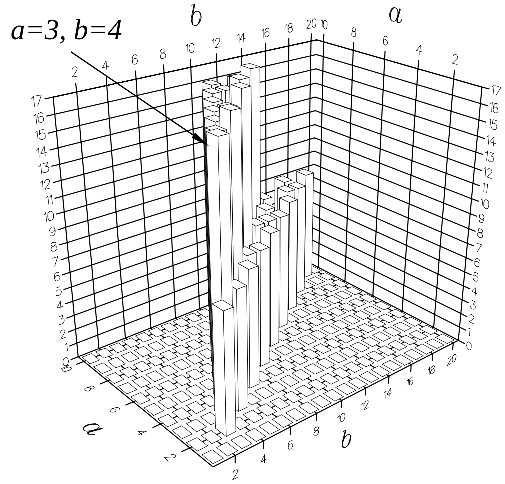

Figure 6: Global Euclidean distance limit for the correctness of the EDT.

2.3 Global Euclidean distance limit and optimal DT coefficients

Our aim now is to determine, whether an optimal pair exists among all valid pairs of DT coefficients. We define optimality here as the smallest integer pair of DT coefficients which guarantees the maximum achievable Euclidean distance limit. Using the results plotted in Figure 6, we could say that, for all pair of DT coefficients such that $a \leq 10$, the pair $(3, 4)$ is a local optimum in the sense that it is the smallest pair of DT coefficients that leads to a (local) maximum Euclidean distance limit (*i.e.*, $\mathcal{R}(3,4) = \sqrt{17}$). In order to extend this result to any pair of DT coefficients, we will use an analytical approach rather than the geometric approach which was used previously.

As suggested in Lemma 4 and Definition 3, an analytical Euclidean distance limit induced by Type 1 errors can be estimated by $R_{\max}(ab)$. Since this limit increases with the values of (a,b), we pointed out earlier that Type 2 error dominates for greater values of DT coefficients. Hence, we will mainly concentrate on an analytical study of Type 2 errors and finally combine the result with those of the previous study of Type 1 errors. The result of this study can be stated as follows.

Theorem 2 *Euclidean distance limit and optimal DT coefficients Considering the chamfer distance $d_{a,b}$ as a discrete distance, the maximal error-free Euclidean distance achievable is $\sqrt{17}$ and the smallest integer pair of DT coefficients that achieves this limit is $(a,b) = (3,4)$.*

We introduce the idea behind the proof of Theorem 2. The approach for deriving an analytical expression of the Euclidean distance limit for Type 2 error (*i.e.*, $\mathcal{R}_2(a,b)$) is made by decomposing the region of the plane (x,y) delimited by the lines $y = x$ and $y = 2x$, by a line $y = \frac{\beta}{\alpha}x$, where (α,β) is an integer pair that matches the conditions for being a pair of DT coefficients (see Figure 7).

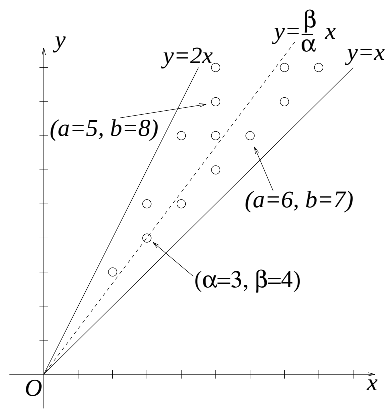

Figure 7: Representation of the valid pairs of DT coefficients for the proof of Theorem 2.

For each such pair (α,β), we characterise a Euclidean distance limit in each sub-region of the plane (x,y) delimited by the lines $y = x$, $y = \frac{\beta}{\alpha}x$ and $y = 2x$. Given a valid pair (α,β), and for any pair of DT coefficients (a,b) different from (α,β), two cases are possible, (i): $\frac{b}{a} < \frac{\beta}{\alpha}$ or (ii): $\frac{b}{a} > \frac{\beta}{\alpha}$. Case (i) includes the valid integer points in the sub-region below $y = \frac{\beta}{\alpha}x$ and above $y = x$, whereas Case (ii) includes the valid integer points in the sub-region above $y = \frac{\beta}{\alpha}x$ and below $y = 2x$. The Euclidean distance limit $\mathcal{R}_2(a,b)$ is to be investigated for the two sub-regions separately and we will refer to this as $\mathcal{R}_{\inf}(\alpha,\beta)$ and $\mathcal{R}_{\sup}(\alpha,\beta)$ for cases (i) and (ii) respectively.

In the example illustrated in Figure 7, $\alpha = 3$, $\beta = 4$. Then, for instance, $\mathcal{R}_2(6,7)$ will include the Euclidean distance limit $\mathcal{R}_{\inf}(3,4)$, since $\frac{7}{6} < \frac{4}{3}$. Similarly, $\mathcal{R}_2(5,8)$ will include the Euclidean distance limit $\mathcal{R}_{\sup}(3,4)$, since $\frac{8}{5} > \frac{4}{3}$.

Hence, given a pair of DT coefficients (a,b), the Euclidean distance limit for Type 2 errors induced by (a,b) (*i.e.*, $\mathcal{R}_2(a,b)$) will result from a combination of all Euclidean distance limits induced by the pairs (α,β) in the following way.

$$\mathcal{R}_2(a,b) = \min\left(\min_{\left\{(\alpha,\beta)/\frac{b}{a}<\frac{\beta}{\alpha}\right\}} \left(\mathcal{R}_{\inf}(\alpha,\beta)\right), \min_{\left\{(\alpha,\beta)/\frac{b}{a}>\frac{\beta}{\alpha}\right\}} \left(\mathcal{R}_{\sup}(\alpha,\beta)\right) \right)$$

Proof:
Given a pair of DT coefficients (a, b) and an integer pair (α, β) such that $0 < \alpha < \beta < 2\alpha$ and $\gcd(\alpha, \beta) = 1$, we consider two cases: *(i)* $\frac{b}{a} < \frac{\beta}{\alpha}$ and, *(ii)* $\frac{b}{a} > \frac{\beta}{\alpha}$. It will become apparent that the equality case represents a Type 1 error and, therefore, as noted earlier, will not be studied here.

We now investigate cases *(i)* and *(ii)* in order to obtain the Euclidean distance limits induced by each case (*i.e.*, $\mathcal{R}_{\inf}(\alpha, \beta)$ and $\mathcal{R}_{\sup}(\alpha, \beta)$, respectively).

(i) Characterisation of $\mathcal{R}_{\inf}(\alpha, \beta)$.
Since, $\frac{b}{a} < \frac{\beta}{\alpha}$ then, $\alpha b < \beta a$. Given $\gamma_1, \gamma_2 \in \mathbb{N}$, we can define the two discrete distances $D_1 = \alpha b + \gamma_1 a + \gamma_2 b$ and $D_2 = \beta a + \gamma_1 a + \gamma_2 b$. In this case, we clearly have $D_1 < D_2$. According to Definition 4, a Type 2 topological error occurs if

$$(\alpha + \gamma_1 + \gamma_2)^2 + (\alpha + \gamma_2)^2 > (\beta + \gamma_1 + \gamma_2)^2 + \gamma_2^2$$

Re-arranging, we obtain,

$$\gamma_2 > \frac{2\gamma_1(\beta \Leftrightarrow \alpha) + \beta^2 \Leftrightarrow 2\alpha^2}{2(2\alpha \Leftrightarrow \beta)} \tag{14}$$

Moreover, from Definition 5, the distance limit induced by the case $\frac{b}{a} < \frac{\beta}{\alpha}$ is, therefore,

$$(\mathcal{R}_{\inf}(\alpha, \beta))^2 = (\beta + \gamma_1 + \gamma_2)^2 + \gamma_2^2 \tag{15}$$

Hence, the first instance for which a Type 2 error occurs is characterised by (γ_1^*, γ_2^*), a pair of positive integers which satisfy the inequality in (14) (*i.e.*, where a Type 2 error occurs) and for which $\mathcal{R}_{\inf}(\alpha, \beta)$ is minimum. Since $\mathcal{R}_{\inf}(\alpha, \beta) > 0$, the minimisation of $\mathcal{R}_{\inf}(\alpha, \beta)$ is equivalent to the minimisation of $(\mathcal{R}_{\inf}(\alpha, \beta))^2$. If we consider the function $\Psi(\gamma_1, \gamma_2) = (\mathcal{R}_{\inf}(\alpha, \beta))^2 = (\beta + \gamma_1 + \gamma_2)^2 + \gamma_2^2$, we can re-write the constrained minimisation problem as follows. Given a valid integer pair (α, β),

$$\begin{array}{ll} & \min \Psi(\gamma_1, \gamma_2) \\ \text{subject to:} & \\ (C_1) & \gamma_2 > \frac{2\gamma_1(\beta - \alpha) + \beta^2 - 2\alpha^2}{2(2\alpha - \beta)} \\ (C_2) & \gamma_1 \geq 0 \\ (C_3) & \gamma_2 \geq 0 \end{array}$$

Now, Ψ is a positive convex quadratic function. Therefore, the global (*i.e.*, unconstrained) minimum of Ψ is reached in $(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}})$ such that $\frac{\partial \Psi}{\partial \gamma_1}(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}}) = 0$ and $\frac{\partial \Psi}{\partial \gamma_2}(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}}) = 0$. Then, Ψ is minimum for $(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}}) = (\Leftrightarrow \beta, 0)$. But, since $\gamma_1^{\text{glob}} = \Leftrightarrow \beta < 0$, the global minimum $(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}})$ does not satisfy constraint (C_2). An analysis of the constraints (see Figure 8) shows that two feasible cases can be considered.

- If $\frac{\beta}{\alpha} < \sqrt{2}$, constraint (C_1) can be removed since, in this case, $(\gamma_1^*, \gamma_2^*) = (0, 0)$ is fully characterised by constraints (C_2) and (C_3), and therefore is the constrained minimum.

- If $\frac{\beta}{\alpha} > \sqrt{2}$, constraint (C_3) becomes redundant. Therefore, in this case, the minimum is obtained using the equalities corresponding to constraints (C_1) and (C_2). Therefore, in this case, $\gamma_1^* = 0$ and γ_2^* is the first integer strictly greater than $\frac{\beta^2 - 2.\alpha^2}{2(2.\alpha - \beta)}$

In both cases, $\mathcal{R}_{\inf}(\alpha, \beta) = \sqrt{(\beta + \gamma_1^* + \gamma_2^*)^2 + \gamma_2^{*2}}$.

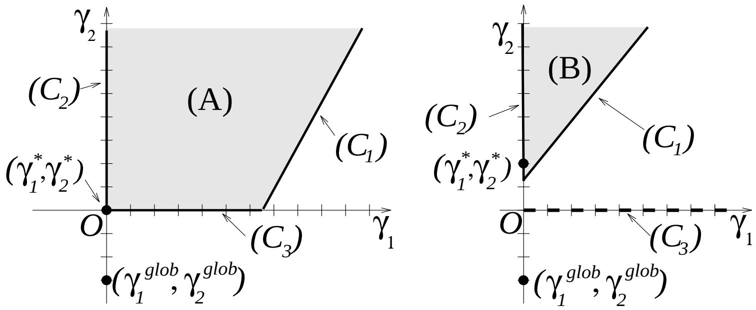

Figure 8: Graphical representation of the constraints (C_1), (C_2) and (C_3). (A) $\frac{\beta}{\alpha} > \sqrt{2}$. (B) $\frac{\beta}{\alpha} < \sqrt{2}$.

(ii) Characterisation of $\mathcal{R}_{\sup}(\alpha, \beta)$.

Let us assume that $\frac{b}{a} > \frac{\beta}{\alpha}$. Following similar analysis to case *(i)*, we have $\Psi(\gamma_1, \gamma_2) = (\alpha + \gamma_1 + \gamma_2)^2 + (\alpha + \gamma_2)^2$. Again, we obtain the following minimisation problem. Given a valid integer pair (α, β),

$$\begin{array}{ll} & \min \Psi(\gamma_1, \gamma_2) \\ \text{subject to:} & \\ (C_1) & \gamma_1 > \frac{2\gamma_2(\beta-2\alpha)+\beta^2-2\alpha^2}{2(\alpha-\beta)} \\ (C_2) & \gamma_1 \geq 0 \\ (C_3) & \gamma_2 \geq 0 \end{array}$$

The global minimum of Ψ is $(\gamma_1^{\text{glob}}, \gamma_2^{\text{glob}}) = (0, \Leftrightarrow\alpha)$ and violates (C_3). Therefore,

- If $\frac{\beta}{\alpha} < \sqrt{2}$, the minimum is obtained for $\gamma_2^* = 0$ and γ_1^* is the first integer strictly greater than $\frac{2\alpha^2-\beta^2}{2(\beta-\alpha)}$.
- If $\frac{\beta}{\alpha} > \sqrt{2}$, $(\gamma_1^*, \gamma_2^*) = (0, 0)$.

In both cases, $\mathcal{R}_{\sup}(\alpha, \beta) = \sqrt{(\alpha + \gamma_1^* + \gamma_2^*)^2 + (\alpha + \gamma_2^*)^2}$.

We can summarise the analytical expressions for the Euclidean distance limits induced by (α, β) in Table 1 (note that in this case, $\lceil x \rceil$ is the smallest integer strictly greater than x).

	$\mathcal{R}_{\inf}(\alpha, \beta) = \sqrt{(\beta + \gamma_1^* + \gamma_2^*)^2 + \gamma_2^{*2}}$	$\mathcal{R}_{\sup}(\alpha, \beta) = \sqrt{(\alpha + \gamma_1^* + \gamma_2^*)^2 + (\alpha + \gamma_2^*)^2}$
$\frac{\beta}{\alpha} < \sqrt{2}$	$\gamma_1^* = 0$, $\gamma_2^* = 0$	$\gamma_1^* = \left\lceil \frac{2\alpha^2-\beta^2}{2(\beta-\alpha)} \right\rceil$, $\gamma_2^* = 0$
$\frac{\beta}{\alpha} > \sqrt{2}$	$\gamma_1^* = 0$, $\gamma_2^* = \left\lceil \frac{\beta^2-2\alpha^2}{2(2\alpha-\beta)} \right\rceil$	$\gamma_1^* = 0$, $\gamma_2^* = 0$

Table 1: Analytical expressions of the distance limits.

Using these results, we can list the values of the limits for the smallest possible values of α and β. Table 2 summarises the Euclidean distance limit values obtained when comparing $\frac{b}{a}$ with $\frac{\beta}{\alpha}$ with the first possible values of α and β.

From the previous results, we can deduce that each possible combination of a, b, α and β creates an increasing sequence when ordered such that $\alpha^2 + \beta^2$ increases (*e.g.*, the expression of $\mathcal{R}_{\inf}(\alpha, \beta)$

α	β	$\mathcal{R}_{\inf}(\alpha,\beta)$	$\mathcal{R}_{\sup}(\alpha,\beta)$
2	3	$\sqrt{17}$	$\sqrt{8}$
3	4	$\sqrt{16}$	$\sqrt{34}$
3	5	$\sqrt{97}$	$\sqrt{18}$
4	5	$\sqrt{25}$	$\sqrt{80}$
4	7	$\sqrt{337}$	$\sqrt{32}$

Table 2: Numerical values Euclidean distance limits deduced from Table 1 for the first instances of (α, β).

when $\frac{\beta}{\alpha} > \sqrt{2}$ leads to the increasing sequence $\sqrt{17}$, $\sqrt{97}$, $\sqrt{337}$, $\cdots$, $\sqrt{\left(\beta + \left\lceil \frac{\beta^2-2.\alpha^2}{2(2.\alpha-\beta)} \right\rceil\right)^2 + \left\lceil \frac{\beta^2-2.\alpha^2}{2(2.\alpha-\beta)} \right\rceil^2}$, $\cdots$). Therefore, all possible Euclidean distance limits will be obtained as soon as we obtain a limit value for any range of $\frac{b}{a}$. Using the two first lines in the previous Table, we deduce that, for $\frac{3}{2} < \frac{b}{a} < 2$, $\mathcal{R}_2(a,b) = \sqrt{8}$, for $\frac{b}{a} = \frac{3}{2}$, $\mathcal{R}_2(a,b) = \sqrt{34}$, for $\frac{4}{3} \leq \frac{b}{a} < \frac{3}{2}$, $\mathcal{R}_2(a,b) = \sqrt{17}$, and for $1 < \frac{b}{a} < \frac{4}{3}$, $\mathcal{R}_2(a,b) = \sqrt{16}$.

As pointed out earlier, $\mathcal{R}_1(a,b)$ increases with the values of the DT coefficients. Hence, clearly $\mathcal{R}(a,b) = \mathcal{R}_2(a,b)$ for any pair of DT coefficients $(a,b) \neq (2,3)$. Now, $\mathcal{R}_2(2,3) = \mathcal{R}_{\sup}(3,4) = \sqrt{34}$, since $\frac{3}{2} > \frac{4}{3}$. From the result of characterisation of Type 1 error, we obtain $\mathcal{R}_1(2,3) = \sqrt{8}$. Therefore, $\mathcal{R}(2,3) = \min(\mathcal{R}_1(2,3), \mathcal{R}_2(2,3)) = \sqrt{8}$.

Therefore, the maximal Euclidean distance achievable is $\max_{\{(a,b)\}} \mathcal{R}(a,b) = \sqrt{17}$. Clearly, the first pair (a,b) which realizes this maximum is $(a,b) = (3,4)$. Hence, Theorem 2 holds. $\square$

In summary, we have extended the results derived from the study presented in Section 2. Theorem 2 states that, for any DT coefficients (a,b) such that $\frac{4}{3} \leq \frac{b}{a} < \frac{3}{2}$, the *topological order* is preserved in any discrete disc of radius D such that $R_{\max}(D) < \sqrt{17}$. In the design of algorithms which require chamfer distances, it is wise to maintain small values of the discrete distances computed. In this context, the minimum DT coefficients for achieving the global upper bound of $\sqrt{17}$ is $(a,b) = (3,4)$. This is emphasised by the fact that the integer pair $(a,b) = (3,4)$ is established as a global optimum for the approximation of Euclidean *distance values*.

3 Nearest Neighbour Problem

In this section, we consider the problem of determining the minimal distance from a grid point to a given set of grid points. More formally, we can formulate the nearest neighbour problem as follows.

(P): Given a reference integer point o and a set of integer points , , find, using integer arithmetic, a point $q \in$, such that $d_{\mathrm{E}}(o,q) = \min_{p \in \Gamma} d_{\mathrm{E}}(o,p)$.

Problem (P) is first reformulated in order to take advantage of the theoretical results presented in Section 2 and to arrive at the Euclidean distance via integer arithmetic. The idea is to characterise two discrete distance bounds, D and D', between which a point q which is the exact solution to (P), will be located. The objective is to minimise distance comparisons in arriving at the exact Euclidean Distance. Using this technique, the search for q is done using the discrete distance $d_{a,b}(.,.)$ rather than the continuous one (*i.e.*, $d_{\mathrm{E}}(.,.)$). From this formulation, we derive an exact algorithm which we express in a pseudo-code. Each step of this algorithm is then described using a graph-theoretic approach and shown to be optimal for the problem in question.

Problem (P) can be equivalently formulated as follows. Given o and , , let q_1 be the point such that:

$$d_{a,b}(o, q_1) = \min_{p \in \Gamma} d_{a,b}(o, p)$$

Following the results derived earlier, q_1 can be considered as an approximation of the exact solution q. Let $D = d_{a,b}(o, q_1)$, then, the solution of (P) is given by the point $q \in$, which is the solution to the following minimisation problem:

$$\min_{p \in \Gamma} d_{\mathrm{E}}(o, p) \text{ subject to } d_{a,b}(o, p) \geq D \;\; \forall\, p \in ,$$

Moreover, $R_{\max}(D)$ gives an upper bound to the distance $d_{\mathrm{E}}(o, q)$. Using the previous characterisation of topological errors (Lemmas 4 and 5), we can define another grid point q_2 as the solution to:

$$R_{\min}(d_{a,b}(o, q_2)) = \min_{\substack{p \in \Gamma \\ R_{\min}(d_{a,b}(o,p)) > R_{\max}(D)}} R_{\min}(d_{a,b}(o, p))$$

In other words, if we note $D' = d_{\mathrm{E}}(o, q_2)$, q_2 is the point in , closest to o such that $R_{\min}(D') > R_{\max}(D)$. Therefore, the solution $q \in$, of (P) is the solution of the following constrained minimisation problem.

$$\min_{p \in \Gamma} d_{\mathrm{E}}(o, p) \text{ subject to } D \leq d_{a,b}(o, p) < D' \;\; \forall\, p \in ,$$

From this new formulation, we can readily implement the following algorithm for the solution of Problem (P).

Algorithm 1

1. Grow a discrete disc centred at o, $\Delta_{a,b}(o, D)$, incrementally ($D = 1, 2, 3 \cdots$) until a point $q_1 \in$, is met.
2. Let $D \leftarrow d_{a,b}(o, q_1)$, $R_{\max} \leftarrow R_{\max}(D)$ and $q \leftarrow q_1$
3. Increase the radius of $\Delta_{a,b}(o, D)$ by 1 ($D \leftarrow D + 1$).
4. If $R_{\min}(D) > R_{\max}$ (*i.e.*, $D = D'$) then stop: q is the solution.
5. If $\exists p \in$, such that $d_{a,b}(o, p) = D$ and $d_{\mathrm{E}}(o, p) < d_{\mathrm{E}}(o, q)$ then, set $q \leftarrow p$.
6. Go to Step 3.

In Step 1, D is obtained as the radius of the smallest discrete disc centred at o and containing a point $q_1 \in$, . Clearly, q_1 is the point of , with the smallest discrete distance to o. In Step 2, $R_{\max}(D)$ is stored and q_1 is considered as an approximation of the solution q. Then, iteratively (Step 3 to 6), the lower bound (*i.e.*, D) is increased and the new points reached by Δ are evaluated by testing whether they are better solutions than the current best q. If it is the case (Step 5), q is updated. Note that the evaluation in Step 5 can also be done using integer arithmetic. This can be carried out by simply using the square of the Euclidean distance or using comparisons on k_{a} and k_{b}. Finally, Step 4 tests for the upper bound D' of the search. Clearly, Algorithm 1 results in the exact solution of the above formulation which is shown to be equivalent to the original formulation of Problem (P). We can therefore state the following Lemma.

Lemma 6 *Given a pair of DT coefficients* (a, b)*, a reference point* o *and a set of points , , Algorithm 1 outputs a point* q *which is the exact solution of Problem* (P) : $\min_{p \in \Gamma} d_{\mathrm{E}}(o, p)$.

Proof:
Direct from the above description of Algorithm 1. □

The algorithm is now presented in a graph theoretic context which allows for optimally performing each step of Algorithm 1. Let $V = \{p = (x_p, y_p)/x_p \in \{x_{\min}, \cdots, x_{\max}\}$ and $y_p \in \{y_{\min}, \cdots, y_{\max}\}\}$, where $x_{\min}$, $x_{\max}$, $y_{\min}$, $y_{\max}$ are such that $o \in V$ and , $\subset V$. Points in V are mapped onto vertices of the grid graph $G = (V, A)$. An arc (p, q) exists in A between two vertices (*i.e.*, integer points) p and q if and only if $d_8(p, q) = 1$ (*i.e.*, if p and q are 8-neighbours). The grid graph G therefore contains all the necessary information about 8-connectivity in V. DT coefficients are directly mapped onto arc lengths in G. The length $l(p, q)$ of the arc $(p, q) \in A$ is such that $l(p, q) = d_{a,b}(p, q)$.

In this context, discrete distance calculations can be performed optimally by the use of well-known shortest path algorithms. More precisely, Dial's adaptation of the shortest path algorithm from Moore [3] is used in Step 1. This algorithm solves the classical shortest problem on a graph first solved by Dijkstra [4] with the use of special data structures (buckets). It iteratively computes the discrete distance label (*i.e.*, length of the shortest path to the root o) of the neighbours of the vertex which have been found to have the smallest such label. In order to optimise this process, it groups vertices with the same distance label in the same bucket. Once all neighbours of a vertex have been updated, the label of the vertex is marked as permanent. Therefore, since we readily obtain a topological ordering of the vertices, a discrete disc of radius D can simply be characterised as the set of permanently labelled vertices as soon as the algorithm attempts to update a vertex such that its new label is strictly greater than D. This algorithm is also known to be optimal in terms of the search time for the shortest path (see [3] for more details). Therefore, in Step 1, we obtain the upper bound q_1 in an optimal number of integer operations.

According to Lemma 3, $R_{\min}(D)$ and $R_{\max}(D)$ can be obtained directly in Steps 2 and 4. Continuing with Dial's shortest path algorithm, it is straightforward to increase the radius of Δ by one. During this operation, only neighbouring vertices of the vertex contained in the current discrete disc Δ have to be considered (*i.e.*, vertices in the D-labelled bucket). We can then readily perform the evaluation described in Step 5 on the newly labelled vertices. It is important to note that Algorithm 1 can be completely and directly (*i.e.*, without any modification) achieved using the above graph theoretic approach. In this context, Lemma 6 is therefore still valid. In other words, the solution found is the exact solution of Problem (P).

We now present an example based on Figure 9. Following Theorem 2, we set $a = 3$ and $b = 4$. The points of , are surrounded by a shaded square. The algorithm goes as follows. A discrete disc Δ is first grown up to q_1 (Step 1). In Figure 9, the points p such that $d_{3-4}(o, p) \leq d_{3-4}(o, q_1)$ are represented by black dots ($\bullet$). Therefore (Step 2), we obtain $D = d_{3-4}(o, q_1) = 47$. $R_{\max}$ can then be calculated, $R_{\max} = d_{\mathrm{E}}(o, q_1) = \sqrt{265}$. Successively (Step 3), D is increased. This leads to check (Step 5) the points labelled $\circ$ ($D = 48$), $\Box$ ($D = 49$), $\otimes$ ($D = 50$) and $\diamond$ ($D = 51$), and not beyond since $R_{\min}(52) = \sqrt{272} > R_{\max}$. Therefore, $D' = d_{3-4}(o, q_2) = 52$ (point q_2 is not actually used, but is indicated by a star ($*$) on Figure 9). For $D = 48$, p_1 is first found to be a solution ($d_{\mathrm{E}}(o, p_1) = \sqrt{250}$). For $D = 49$, p_2, is found to be closer to o than p_1. No other point from , are to be investigated, therefore, $q = p_2$ is the exact solution with $d_{\mathrm{E}}(o, q) = \sqrt{245}$.

In summary, Algorithm 1 results in the exact solution of Problem (P). Given a point o and using discrete distance calculations, Algorithm 1 solves exactly the problem of minimising the Euclidean distance between o and a given set of points , . Since the search for the solution is done using discrete distances, we can use integer arithmetic. Moreover, the optimality of Dial's shortest path algorithm guarantees the optimality of Algorithm 1. Clearly, the complexity of this algorithm derives directly

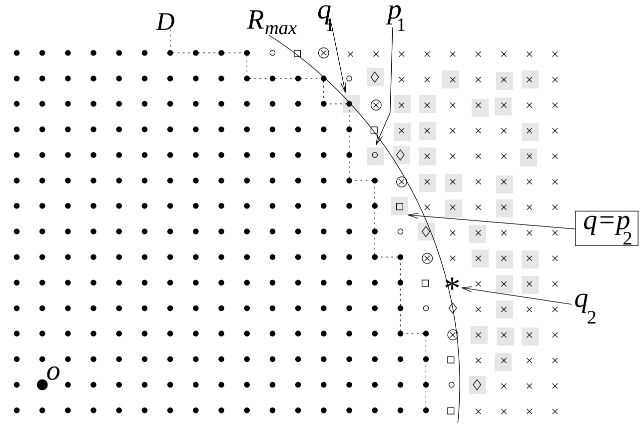

Figure 9: An example application of Algorithm 1.

from that of Dial's shortest path algorithm which is linearly dependent on the number of arcs of the grid graph. Moreover, as we use the 8-neighbourhood, this number is asymptotically equivalent to the number of vertices (*i.e.*, $m \leq 8n$, where $n = |V|$ and $m = |A|$ indicate the number of vertices and arcs in G respectively). As noted earlier, since this algorithm performs an ordered search, the first approximation of the solution (namely q_1) will be found (Step 1) in a minimum number of integer operations. Furthermore, the theoretical results in Section 2 guarantee that the number of points for which the real distance is compared is minimum. Therefore, we can state that Algorithm 1 solves Problem (P) optimally. Note that Euclidean distance comparisons are performed on the squares of these distances to maintain integer arithmetic throughout.

4 Conclusion

In this paper, the problem of approximating continuous distances by discrete ones was considered. We considered the Euclidean distance in the continuous space whereas the chamfer distance based on 3×3 masks (*i.e.*, using the DT coefficients (a, b)) was used in the discrete space. We formally characterised the topological errors which occur during the mapping of distances from the discrete to the continuous space. Distance limits up to which these errors are guaranteed not to occur were derived for any pair of DT coefficients. Among all DT coefficients, an optimal integer pair was characterised and shown analytically to correspond to a global optimum. As by-product of this study, we obtained results which give, without the need of enumeration, all possible decompositions of a discrete distance value into a combination of moves on a shortest path on the grid $(k_{\mathrm{a}}, k_{\mathrm{b}})$. We also obtained a result which allows for the direct computation of the maximal and minimal Euclidean distances induced by such decompositions for any discrete distance value. We formalise the results by obtaining optimal DT coefficients in terms of the topological ordering they induce on the discrete grid, in contrast with optimality in terms of the approximation error of continuous distance values. These results also allow for further understanding of errors made when approximating Euclidean distances by discrete ones.

As an application, we presented an optimal algorithm for solving the nearest neighbour problem based on Euclidean distances using integer arithmetic This problem is representative of numerous problems encountered in the field of digital image processing. Finally, we briefly introduced the graph-theoretical context which creates a robust context for solving discrete optimisation problems. Such an approach has been successfully applied (*e.g.*, [9, 12]) and warrants further investigation.

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