PERFORMANCE OF BLIND DISCRETE SOURCE SEPARATION

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ABSTRACT

The aim of this paper is two-fold. In a first part, we investigate the ultimate performances of source separation in the case of BPSK and 4-PSK sources. These ultimate bounds are computed by looking first for the most favourable mixing matrices, for various SNRs, number of sources and number of sensors, and then by calculating the associated error probabilities. In a second part, we compare the behaviour of two 4-QAM source separation algorithms. One is based on the Constant Modulus (CM) property of these sources, and the other is based on the constance of their fourth power. The goal here is to see what kind of improvement brings the knowledge of the source distribution.

1 INTRODUCTION

Because of the impressive increase in the number of subscribers, improvement on wireless communications has become of prior importance. The use of spatial diversity granted by multiple antennas is one of the means that must be resorted to. Blind source separation is one way of taking advantage of spatial diversity, without the help of learning sequences. Tanks to blind techniques, the transmission rate could then be increased in GSM or UMTS mobile systems. Other applications are not reported here.

Many algorithms have been proposed to solve blind source separation using various criteria such as the independence of the sources [2] [1] or their constant modulus property [6] [5]. But the discrete character of communication signals has been exploited in the blind context only for the last few years [4] [3]. In this paper, we are interested in the case where the source distribution is discrete and known.

Our goal is two-fold. First, we investigate the ultimate bounds of BPSK or 4-PSK source separation (even when there are more sources than sensors). These bounds are computed by looking for the most favourable mixing matrix in each configuration and by finding the associated error probability.

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Second, we present a new analytical solution for the separation of 4-QAM signals based on their distribution. Then we compare the performances of this new algorithm with the CM analytical algorithm of [6].

This paper is organized as follows. In section 2, ultimate bounds are computed in the case of one and two sensors, and different numbers of sources. In section 4, we present our analytical solution for 4-QAM sources. Finally, we investigate performances of this algorithm with computer results.

Notation

As far as we know, no work has been carried out to find the ultimate bounds in the case of discrete sources with known distribution, ignoring the modulation memory.

In order for the bounds to be insensitive to the mixing matrix, the best mixture has been considered, in terms of error probability. In addition, the number of sources and their distribution have been assumed to be known, hence the name of ultimate bounds.

Assume the following baseband model :

$$\mathbf{y} = A\mathbf{x} + \mathbf{w} \tag{1}$$

where $\mathbf{y}$ is the output of an array of K sensors impinged by P independent unit-variance sources represented by vector $\mathbf{x}$, and $\mathbf{w}$ is a white gaussian noise $\mathcal{N}(0, \sigma^2)$ independent of $\mathbf{x}$. Without loss of generality, we assume that each entry of the vector $A\mathbf{x}$ has a unit variance, *i.e.*, $|a_{k1}|^2 + \cdots + |a_{kP}|^2 = 1$.

2 BPSK SOURCE SEPARATION

2.1 Case $K = 1$ sensor, $1 \leq P \leq 3$ sources

Now, we present some results for BPSK sources. The sensor output can be written :

$$y = \sum_{i=1}^{P} a_i x_i + w$$

- $P = 1$, $a_i \in \mathbb{C}$ $\quad \begin{cases} a_1 &= exp\,(i\Phi) \\ P_\epsilon &= \frac{1}{2}\mathrm{erfc}\left(\frac{1}{\sigma\sqrt{2}}\right) \end{cases}$
- $P = 2$ *sources*, $a_i \in \mathbb{C}$
  Since BPSK signals are real, it is obvious that the

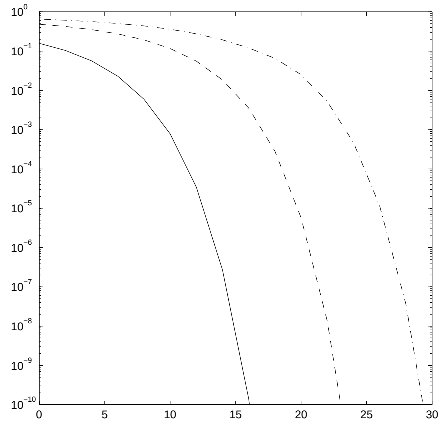

Figure 1: Ultimate bounds of BPSK source separation with $K = 1$ sensor, $1 \leq P \leq 3$ sources and when A is real

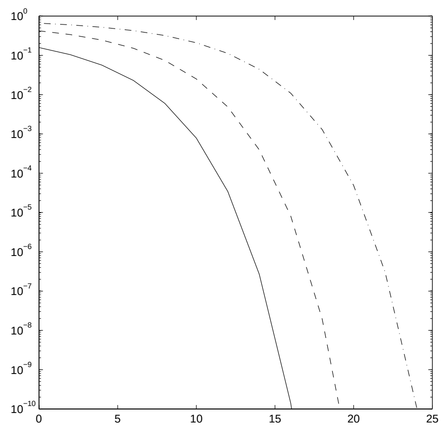

Figure 2: Ultimate bounds of BPSK source separation with $K = 1$ sensor, $1 \leq P \leq 3$ sources and when A is complex

best case is obtained when x_1 and x_2 are on orthogonal axes, that is :

$$\left\{ \begin{array}{lcl} A & = & \left(\frac{\sqrt{2}}{2} e^{i\Phi} \;\; \frac{\sqrt{2}}{2} e^{i\left(\Phi \pm \frac{\pi}{2}\right)} \right) \\ P_\epsilon & = & \operatorname{erfc}\left(\frac{1}{2\sigma}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\frac{1}{2\sigma}\right) \end{array} \right.$$

- $P = 2$ *sources,* $a_i \in I\!R$. We have:
  $P_\epsilon = \frac{1}{2}\operatorname{erfc}\left(\frac{a_{12}}{\sigma\sqrt{2}}\right) + \frac{1}{4}\operatorname{erfc}\left(\frac{a_{11}-a_{12}}{\sigma\sqrt{2}}\right)$

- $P = 3$ *sources,* $a_i \in I\!R$. The error probability takes the possible two following forms.
  Case : $0 \leq a_3 \leq a_2 \leq a_1$ and $a_1 \geq a_2 + a_3$

$$P_\epsilon = \frac{1}{4}\left[2\operatorname{erfc}\left(\frac{a_3}{\sigma\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{a_2-a_3}{\sigma\sqrt{2}}\right) + \frac{1}{2}\operatorname{erfc}\left(\frac{a_1-a_2-a_3}{\sigma\sqrt{2}}\right)\right]$$

Case : $0 \leq a_3 \leq a_2 \leq a_1$ and $a_1 \geq a_2 + a_3$

$$P_\epsilon = \frac{1}{8}\left[2\operatorname{erfc}\left(\frac{a_3}{\sigma\sqrt{2}}\right) + 2\operatorname{erfc}\left(\frac{a_1-a_2}{\sigma\sqrt{2}}\right) + \frac{1}{2} + 2\operatorname{erfc}\left(\frac{a_2-a_3}{\sigma\sqrt{2}}\right) + \frac{1}{2}\operatorname{erfc}\left(\frac{a_2+a_3-a_1}{\sigma\sqrt{2}}\right)\right]$$

By exhaustive search, the error probability has been minimized in both cases for various SNRs. The second case is the best at low SNRs and the first is the most attractive at high SNRs.

- $P = 3$ *sources,* $(a1, a2, a3) \in \mathbb{C}$
  This case is not completly solved because neither the decision areas nor the associated probability errors are easy to compute. Nevertheless, it can be shown that the maximization of the minimal distance between signal values leads to $\sqrt{6}\,(a_1, a_2, a_3) = (2, i, 1)$; this is the optimal solution for high SNRs. Because of symmetry, the best mixing matrices have the following structure : $A = (\; a_1 \; ia_2 \; a_3 \;)$ where a_1, a_2 and a_3 are real. On the other hand, P_ϵ is equal to :

$$\begin{aligned} P_\epsilon = \; & \operatorname{erfc}\left(\frac{a_2}{\sigma\sqrt{2}}\right) + \operatorname{erfc}\left(\frac{a_3}{\sigma\sqrt{2}}\right) + \frac{1}{2}\operatorname{erfc}\left(\frac{a_1-a_2}{\sigma\sqrt{2}}\right) \\ & -\frac{1}{2}\operatorname{erfc}\left(\frac{a_2}{\sigma\sqrt{2}}\right)\operatorname{erfc}\left(\frac{a_3}{\sigma\sqrt{2}}\right) \\ & -\frac{1}{4}\operatorname{erfc}\left(\frac{a_2}{\sigma\sqrt{2}}\right)\operatorname{erfc}\left(\frac{a_1-a_3}{\sigma\sqrt{2}}\right) \end{aligned}$$

The results obtained by minimizing P_ϵ with respect to the real triplet (a_1, a_2, a_3) are presented in figures 1 and 2.

2.2 Case $K = 2$ sensors, $1 \leq P \leq 4$ sources

Here one can use the previous results obtained in the single sensor case stating that each row of the best mixing matrix is the best single sensor receiver, bearing in mind that the mixing matrix must be full rank.

- $P = 1$ *source*
  In this case, $a_{11} = exp\,(i\Phi_1)$ and $a_{21} = exp\,(i\Phi_2)$ and the error probability is :

$$P_\epsilon = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{\sigma}\right)$$

- $P = 2$ *sources*
  Here, the best case is given by a diagonal matrix with unit modulus elements so that :

$$P_\epsilon(x_1, x_2) = \operatorname{erfc}\left(\frac{1}{\sigma\sqrt{2}}\right) - \frac{1}{4}\,erfc^2\left(\frac{1}{\sigma\sqrt{2}}\right)$$

- $P = 3$ *sources and* $A \in \mathbb{C}$
  Since BPSK signals are real, one supposes that the best mixture has the following structure :

$$A = \begin{pmatrix} a_1 & ia_2 & 0 \\ 0 & ia_3 & a_4 \end{pmatrix}$$

where a_1, a_2, a_3 and a_4 are real positive coefficients. The associated error probability is :

$$\begin{aligned} P_\epsilon = \; & 1 - \frac{1}{8}\operatorname{erfc}\left(-\frac{a_1}{\sigma\sqrt{2}}\right)\operatorname{erfc}\left(-\frac{a_4}{\sigma\sqrt{2}}\right) \\ & \operatorname{erfc}\left(-\frac{a_2+a_3}{\sigma\sqrt{2(a_2+a_3)}}\right) \end{aligned}$$

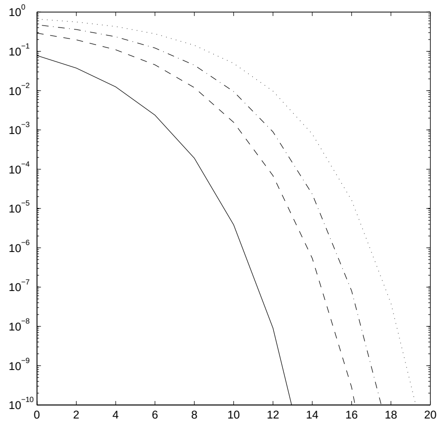

Figure 3: Ultimate bounds of BPSK source separation with $K = 2$ sensors, $1 \leq P \leq 4$ sources and A is complex

- *$P = 4$ sources and $A \in \mathbb{C}$*
  Here, the best case is obviously given by :

$$A = \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & a_4 \end{pmatrix}$$

  where $a_1 = \frac{\sqrt{(2)}}{2} e^{i\Phi_1}$, $a_2 = i a_1$, $a_3 = \frac{\sqrt{(2)}}{2} e^{i\Phi_2}$ and $a_4 = i a_3$. The associated error probability is :

$$P_\epsilon \quad = \quad 1 - \tfrac{1}{16} \mathrm{erfc}^4 \left(-\tfrac{1}{2\sigma} \right)$$

The results are shown in figure 3.

3 4-PSK SOURCE SEPARATION

This case has not been completly studied. We only investigated the real mixing matrix configurations.

3.1 Case $K = 1$ sensor, $1 \leq P \leq 2$ sources

- *$P = 1$ source* :

$$P_\epsilon = \mathrm{erfc}\left(\tfrac{1}{2\sigma}\right) - \tfrac{1}{4} \mathrm{erfc}^2 \left(\tfrac{1}{2\sigma}\right)$$

- *$P = 2$ sources, $(a_1,\ a_2) \in \mathbb{R}^2$*
  With no loss of generality, we assume that $0 \leq a_2 \leq a_1$. The error probability is then equal to :

$$\begin{aligned} P_\epsilon \quad = \quad & \tfrac{1}{16} \Big[16 \, \mathrm{erfc}\left(\tfrac{a_2}{2\sigma}\right) + 8 \, \mathrm{erfc}\left(\tfrac{a_1 - a_2}{2\sigma}\right) \\ & -4 \, \mathrm{erfc}^2\left(\tfrac{a_2}{2\sigma}\right) - \mathrm{erfc}^2\left(\tfrac{a_1 - a_2}{2\sigma}\right) \\ & -4 \, \mathrm{erfc}\left(\tfrac{a_2}{2\sigma}\right) \mathrm{erfc}\left(\tfrac{a_1 - a_2}{2\sigma}\right) \Big] \end{aligned}$$

  and has been minimized by a combinatorial optimization.

The results are presented in Figure 4

3.2 Case $K = 2$ sensors, $1 \leq P \leq 2$ sources

- *One source.* The error probability is equal to :

$$P_\epsilon = \mathrm{erfc}\left(\tfrac{1}{\sigma}\right) - \tfrac{1}{4} \, \mathrm{erfc}^2 \left(\tfrac{1}{\sigma}\right)$$

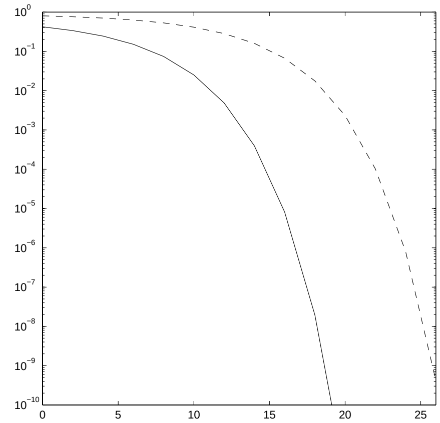

Figure 4: Ultimate bounds of 4-PSK source separation with $K = 1$ sensor, $1 \leq P \leq 2$ sources and A is real.

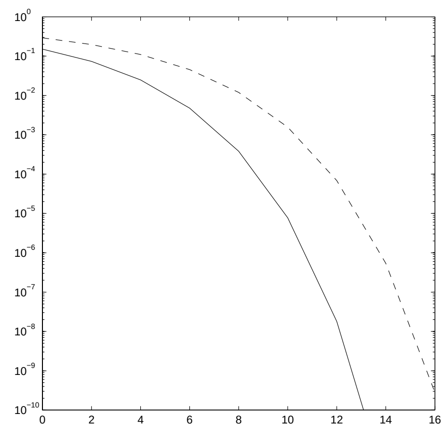

Figure 5: Ultimate bounds of 4-PSK source separation with $K = 2$ sensors, $1 \leq P \leq 2$ and A is complex.

- *Two sources.* The best mixing matrix is diagonal with unit modulus entries; the error probability is:

$$P_\epsilon = \mathrm{erfc}\left(-\tfrac{1}{\sigma\sqrt{2}}\right) - \tfrac{1}{4} \, \mathrm{erfc}^2 \left(-\tfrac{1}{\sigma\sqrt{2}}\right)$$

The results are reported in figure 5

4 ANALYTICAL SOURCE SEPARATION WITH 4-QAM SIGNALS

The analytical solution to 4-QAM source separation is based on the fact that the fourth power of the elements of this constellation is constant. In essence, it has the same structure as the analytical CMA presented in [6] and may be considered as a generalization to the **constant fourth power** signals.

Rewrite the narrow band reception model in block form:

$$Y = X \, A + W. \tag{2}$$

Denote $U \, \Sigma \, V^\dagger$ the SVD of Y. The separation problem consists of finding a matrix B such that the columns of $\hat{X} = UB$ are close to 4-QAM signals. In the noiseless case, this property means that:

$$(u_n^t\,\mathbf{b}_p)^4 = 1,\ \forall p,\ \forall n \qquad (3)$$

where $\mathbf{b}_p$ is the p^{th} column of B, and $U = [u_1, \cdots, u_N]^t$.

Let ${}^4\mathbf{z}$ be the non redundant vectorization of the symetric fourth order tensors associated with any vector $\mathbf{z}$, equation (3) is then equivalent to :

$${}^4U\,{}^4\mathbf{b}_p = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \qquad (4)$$

where ${}^4U = \left[{}^4u_1, \cdots, {}^4u_N\right]^t$, and ^{t} denotes transposition (without conjugation). Now, if we multiply equation (4) by the appropriate matrix H, we obtain, with obvious notation:

$$H\,{}^4U\,{}^4\mathbf{b}_p = \begin{bmatrix} \sqrt{N} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad {}^4U_H\,{}^4\mathbf{b}_p = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \qquad (5)$$

Therefore ${}^4\mathbf{b}_p$ is in the kernel of ${}^4U_H$.

In the noiseless case, the dimension of this kernel is equal to the number of 4-QAM signals present in the mixture Y. Let $\{\mathbf{e}_1, \cdots, \mathbf{e}_2\}$ be a basis of $Ker({}^4U_H)$ and note that the P solutions b_p of equation (3) form a basis of $Ker({}^4U_H)$ too. Thus the following relation between $\mathbf{e}_i$ and $\mathbf{b}_p$ holds

$$\left\{\begin{array}{ccl} \mathbf{e}_1 & = & \sum_{i=1}^{P} \alpha_{1i}\,{}^4\mathbf{b}_i \\ & \vdots & \\ \mathbf{e}_P & = & \sum_{i=1}^{P} \alpha_{Pi}\,{}^4\mathbf{b}_i \end{array}\right. \qquad (6)$$

Now let $unvec_2^4$ be an operator such that $unvec_2^4({}^4\mathbf{z}) = {}^2\mathbf{z}\,{}^2\mathbf{z}^t$ where ${}^2\mathbf{z}$ is the non redundant vectorization of $\mathbf{z}\,\mathbf{z}^t$. System (6) is then equivalent to :

$$\left\{\begin{array}{ccl} E_1 & = & \sum_{i=1}^{P} \alpha_{1i}\,{}^2\mathbf{b}_i\,{}^2\mathbf{b}_i^t \\ & \vdots & \\ E_P & = & \sum_{i=1}^{P} \alpha_{Pi}\,{}^2\mathbf{b}_i\,{}^2\mathbf{b}_i^t \end{array}\right. \qquad (7)$$

which can be rewritten as:

$$\left\{\begin{array}{ccl} E_1 & = & {}^2B\,D_1\,{}^2B^t \\ \ldots & = & \ldots \\ E_P & = & {}^2B\,D_P\,{}^2B^t \end{array}\right. \qquad (8)$$

where $E_p = unvec_2^4(\mathbf{e}_p)$ and ${}^2B = \left[{}^2\mathbf{b}_1, \cdots, {}^2\mathbf{b}_P\right]$.

Therefore the search for B is equivalent to simultaneously diagnalize P matrices E_p. If $P = 2$ or in the noiseless case, the joint diagonalization of E_1 and E_2 is straightforward and suffices to compute ${}^2B$. However, it requires E_1 to be regular. Yet, in the present case, they are both singular; the following solution is proposed : ${}^2B = Q_2^{-1}Q_1^{-}$ where $Q_1^{-}E_1Q_1 = \Lambda_1$ is the EVD of E_1, Λ_1 is $P \times P$ diagonal, Q_1^{-} is the column pseudo-inverse of Q_1, and $Q_2^{-1}\Lambda_1(Q^{-}E_2Q)^{-1}Q_2$ is $P \times P$ diagonal.

Once ${}^2B$ has been obtained its P columns give all the solutions to the equation (3), $\mathbf{b}_p$ being given by the rank-one approximation of $unvec({}^2\mathbf{b}_p)$.

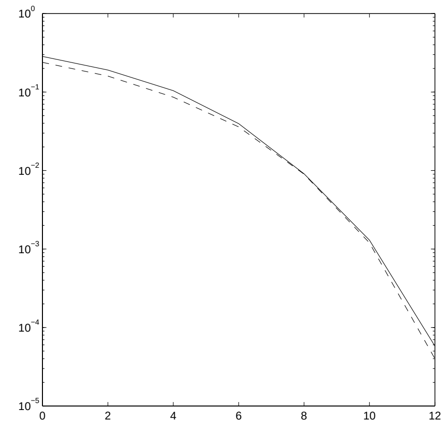

Figure 6: Bit Error Rates of both analytical solutions for various SNRs

5 ALGORITHMS COMPARISON WITH COMPUTER RESULTS

The simulations have been made using an array with $K = 2$ sensors. We used $P = 2$ 4-PSK sources whose angles of arrival were $\theta_1 = -30°$ and $\theta_2 = 30°$. Both algorithms were tested with data length $N = 400$, over 200 trials and for various SNRs.

The results are plotted in figure 6. The performances are very close. Hence surprisingly, the knowledge of the distribution doesn't bring significant improvement in the algorithm. This needs to be elucidated.

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