On the Capacity of Asynchronous CDMA Systems

Laura Cottatellucci
Institut Eurecom
Sophia Antipolis, France
Email: laura.cottatellucci@eurecom

Ralf R. Müller
Norwegian Univ. of Science and Technology
7491 Trondheim Norway
Email: mueller@iet.ntnu.no

Mérouane Debbah
Supelec
Paris, France
Email: merouane.debbah@supelec.fr

***Abstract*— The total capacity per chip constrained to a given chip pulse waveform of asynchronous code division multiple access (CDMA) channels with random spreading and subject to frequency-flat fading is investigated in the large system limit. The analysis in terms of signal to interference and noise ratio (SINR) is extended to CDMA systems with linear minimum mean square error (MMSE) detectors. The system behaviour is completely described by a positive function $\eta(f)$ that can be interpreted as the spectrum of the multiuser efficiency. Both total capacity per chip and SINR of linear MMSE detectors can be expressed in terms of the spectrum of the multiuser efficiency. A simple relation between the total capacities per chip of asynchronous CDMA systems with modulation based on sinc pulse waveforms and of synchronous CDMA systems is derived.**

I. Introduction

The fundamental limits of synchronous CDMA systems have been thoroughly studied by modelling the spreading sequences by random sequences in [1], [2], [3]. However, this analysis is focused on synchronous CDMA systems while the assumption of synchronism is not realistic for the uplink of a CDMA system. Therefore, it is of theoretical and practical interest to extend the analysis of CDMA systems with random spreading to the asynchronous case.

The analysis of asynchronous CDMA systems limited to symbol asynchronous but chip synchronous signals, i.e. signals whose time delays are multiple of the chip interval, is in [4], [5]. In [4] the performance of a linear MMSE detector with infinite observation window is proven to be equivalent to the performance of a synchronous CDMA system. The performance degradation of linear detectors with finite observation windows has been analyzed in [5]. In [6] linear MMSE detectors for asynchronous CDMA systems with modulation based on an ideal Nyquist sinc function (bandwidth equal to half of the chip rate or Nyquist rate R_{Nyq}) are shown to be equivalent in terms of performance to linear MMSE detectors for synchronous CDMA systems. The effects of chip asynchronism on the performance of linear multistage detectors have been object of study in [7], [8]. Asynchronous CDMA systems with multistage detectors at the receiver and modulation based on chip pulse waveforms with bandwidth not greater than the Nyquist rate have the same asymptotic performance as the correspondent synchronous systems. Furthermore, the performance is independent of the time delay distribution. Increasing the bandwidth of the chip waveform beyond the Nyquist rate, the system performance changes substantially. It depends on the time delay distribution and the equivalence between synchronous and asynchronous systems does not hold [7]. The impact of pulse shaping and the performance loss of linear multistage detectors due to the use of suboptimal statistics in asynchronous CDMA systems is in [8]. The optimum design criterion for synchronous CDMA systems based on the principle of interchip interference free pulses (Nyquist criterion) does not hold for asynchronous systems. Then, for asynchronous CDMA systems the analysis of general waveforms becomes fundamental.

From a technical point of view, the large system performance analysis in [7], [8] is based on recursive expressions of the limit eigenvalue moments of the system covariance matrix $\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H$ (here $\boldsymbol{\mathcal{H}}$ is the transfer matrix of the system) while to derive the total capacity per chip of a large CDMA system or its SINR at the output of a linear MMSE detector the knowledge of the limit eigenvalue distribution is required (see [1], [2], [3]). Thus, the results in [7], [8] do not enable a large system analysis of the effects of asynchronism on these two relevant performance measures. In this work we investigate the fundamental limits of asynchronous CDMA systems in terms of both total capacity per chip constrained to a chip pulse waveform and the limit SINR of linear MMSE detectors.

Due to space restriction in this work we consider (i) CDMA systems with modulation based on chip pulse waveforms with bandwidth not greater than the Nyquist rate and any set of time delays or (ii) CDMA systems using chip pulse waveforms with bandwidth greater than the Nyquist rate and uniform time delay distribution. The pulse waveform is completely general and it is not required to satisfy the Nyquist criterion. For the general case of CDMA with any time delay distribution the interested reader can refer to [9]. In both case (i) and case (ii) the behaviour of large CDMA systems with linear MMSE detectors is completely described by a positive scalar function $\eta(N_0, f)$ of the power spectral noise N_0 and the frequency f. This function can be interpreted as the spectrum of the multiuser efficiency of the linear MMSE detector at a given level of power noise N_0.

In [2] it was shown that the limiting interference effects under linear MMSE detection can be decoupled in large synchronous CDMA systems using random spreading sequences. The level of interference that can be ascribed to an interferer k is referred to as effective interference. Beside the decoupling effects on interferers as in synchronous CDMA systems the large system analysis of asynchronous systems shows an additional decoupling effect in frequency such that the concept of spectrum of the effective interference can be introduced. The effective interference at frequency f of a user k on the user of interest is the level of interference that can be attributed to the component of the signal of user k at frequency f in the

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detection of the user of interest.

Furthermore, we consider asynchronous CDMA systems with number of transmitted symbols per chip β and modulation based on a sinc function with bandwidth proportional to the Nyquist rate by a positive real factor γ, i.e. $B = \gamma R_{\rm Nyq}$. We show that a linear MMSE detector performs as well as in a synchronous system with modulation based on square root Nyquist pulses and system load $\beta' = \frac{\beta}{\gamma}$. This property implies the possibility to trade degrees of freedom in the frequency domain provided by the bandwidth of the chip pulse waveform against the degrees of freedom in the time domain provided by the spreading factor N.

For CDMA systems (i) and (ii) also the total capacity per chip can be expressed as a function of the spectrum of the multiuser efficiency. An explicit expression for the constrained total capacity is provided for modulation based on a sinc function with bandwidth $B = \gamma R_{\rm Nyq}$. The constrained total capacity per chip of a system with load β is related to the capacity of synchronous systems. More specifically, the capacity is γ times the total capacity per chip of a synchronous system with system load $\beta' = \frac{\beta}{\gamma}$. In synchronous CDMA systems the maximum total capacity per chip constrained to a given bandwidth $\gamma R_{\rm Nyq}$ is achieved by modulation based on square root Nyquist functions and it is constant for any $\gamma \in [1, +\infty]$. In contrast, the capacity constrained to a sinc pulse increases with the bandwidth in asynchronous CDMA systems and asynchronous systems outperform the synchronous ones. The gap between the spectral efficiency of a (synchronous or asynchronous) CDMA system with ideal Nyquist sinc pulse ($\gamma = 1$) and the spectral efficiency of an asynchronous system using a sinc pulse with any $\gamma > 1$ vanishes asymptotically as the system load tends to infinity, i.e. as $\beta \to \infty$ at a constant level of the energy per bit per noise level $\frac{E_b}{N_0}$.

Due to space restriction the proofs of the theorems are omitted in this article. The interested reader can refer to [9].

II. System Model

Let us consider an asynchronous CDMA system with K users and spreading factor N in an uplink fading channel impaired by additive white Gaussian noise (AWGN). Then, $\beta = \frac{K}{N}$ is the system load and the signal received at the base station, in complex base-band notation, is given by

$$y(t) = \sum_{k=1}^{K} a_{kk} s_k(t - \tau_k) + n(t) \qquad t \in [-\infty, +\infty].$$

Here, a_{kk} is the received signal amplitude of user k; τ_k is the time delay of user k; $n(t)$ is a zero mean white, complex Gaussian process with two-sided power spectral density N_0; and $s_k(t)$ is the spread signal of user k. We have

$$s_k(t) = \sum_{m=-\infty}^{+\infty} b_k[m] c_k^{(m)}(t),$$

where $b_k[m]$ is the $m^{\rm th}$ transmitted symbol of user k and

$$c_k^{(m)}(t) = \sum_{u=0}^{N-1} s_{k,m}[u] \phi(t - mT_s - uT_c)$$

is its spreading waveform at time m. Here, $\boldsymbol{s}_{k,m}$, is the spreading sequence of user k in the $m^{\rm th}$ symbol interval with elements $s_{k,m}[u]$, $u = 0, \ldots, N-1$; T_s and $T_c = \frac{T_s}{N}$ are the symbol and chip periods, respectively.

The users' symbols $b_k[m]$ are uncorrelated and identically distributed random variables with $\mathrm{E}\{|b_k[m]|^2\} = 1$ and $\mathrm{E}\{b_k[m]\} = 0$. The elements of the spreading sequences $s_{k,m}[u]$ are assumed to be i.i.d. random variables with $\mathrm{E}\{|s_{k,m}[u]|^2\} = \frac{1}{N}$ and $\mathrm{E}\{s_{k,m}[u]\} = 0$.

The chip waveform $\phi(t)$ is bandlimited with bandwidth B, unit energy, and Fourier transform $\Phi(f)$. Thanks to the statistical properties of the spreading sequences, the average energy of the signature waveform is also unit.

At the front-end the base band signal is processed by a lowpass filter with lowpass band $B_{\rm FE} = \frac{r}{2T_c}$ and $r \geq 2BT_c$. Then, the chip pulse waveform at the output of the low pass filter is still $\Phi(f)$. The filter output is sampled at rate $\frac{r}{T_c}$ such that the conditions of the sampling theorem are satisfied. With this choice of the front-end the sampled signal provides sufficient statistics and the discrete-time noise is still white with zero mean and variance $\sigma^2 = \frac{N_0 r}{T_c}$.

The discrete-time signal at the front-end output is given by

$$y[p] = \sum_{k=1}^{K} a_k \sum_{m=-\infty}^{+\infty} b_k[m] \sum_{u=0}^{N-1} s_{k,m}[u] \phi\Big(\frac{p}{r}T_c - \tau_k - (u + mN)T_c\Big) + n[p] \tag{1}$$

with $p \in \mathbb{Z}$ and $n[p]$ the discrete-time, complex-valued noise.

Throughout this work we assume that the filtered chip pulse waveform $\phi(t)$ is much shorter than the symbol waveform, i.e. $\phi(t)$ becomes negligible for $|t| > t_0$ and $t_0 \ll T_s$. This is usually verified in the systems with large spreading factor, which we are considering. Thus, we can neglect the intersymbol interference. Then, given the time delay τ_k the virtual spreading sequence of user k for the transmitted symbol m spans the symbol intervals m and $m+1$ and it is a $2Nr$-dimensional vector given by

$$\boldsymbol{v}_{km} = \boldsymbol{\Phi}_k \boldsymbol{s}_{km}$$

where $\boldsymbol{s}_{km} = (s_{km}[0] \ldots s_{km}[N-1])^T$ and $\boldsymbol{\Phi}_k$ is a $2Nr \times N$ matrix taking into account the effects of the pulse shape and the time delay of user k. The matrix $\boldsymbol{\Phi}_k$ is of the form

$$\boldsymbol{\Phi}_k = \left[\begin{array}{ccc} \boldsymbol{0}_{k,0}^T & \boldsymbol{C}_{\phi,r}^T \left(\tau_k - \lfloor \frac{\tau_k}{T_c} \rfloor T_c\right) & \boldsymbol{0}_{k,1}^T \end{array} \right]^T \tag{2}$$

where $\boldsymbol{0}_{k,0}$ and $\boldsymbol{0}_{k,1}$ are matrices of dimensions $\lfloor \frac{r\tau_k}{T_c} \rfloor \times N$ and $\left(N - \lfloor \frac{r\tau_k}{T_c} \rfloor\right) \times N$, respectively, with zero elements; $\boldsymbol{C}_{\phi,r}(\tau_k)$ is an r-block-wise circulant matrix[1] of order N defined by

$$\boldsymbol{C}_{\phi,r}(\tau) \triangleq \boldsymbol{C}\left(\phi(x,\tau), \phi\left(x, \tau - \frac{T_c}{r}\right), \ldots, \phi\left(x, \tau - \frac{(r-1)T_c}{r}\right)\right), \tag{3}$$

with

$$\phi(x,\tau) \triangleq \frac{1}{T_c} \sum_{s=-\infty}^{+\infty} \mathrm{e}^{j2\pi \frac{\tau}{T_c}(x+s)} \Phi^*\left(\frac{j2\pi}{T_c}(x+s)\right). \tag{4}$$

The zero matrices take into account the fact that we neglect the the useful signal outside the interval $[mT_s + \tau_k, (m+1)T_s + \tau_k]$. For K and N finite, the circulant matrix $\boldsymbol{C}_{\phi,r}(\tau)$ as defined in (3) approximates the matrix directly derivable from (1), which is Toeplitz. Furthermore, the two matrices are

[1] An r-blockwise circulant matrix of order N is an $rN \times N$ matrix of N block rows of dimensions $r \times N$ such that each block is obtained by circularly right shifting of the previous block. It is completely characterized by the r Fourier transforms of each of the rows.

asymptotically equivalent in terms of spectral distribution (see e.g. [10]).

Let $\boldsymbol{S}[m]$ be the $2rN \times K$ matrix of virtual spreading, i.e. $\boldsymbol{S}[m] = (\boldsymbol{\Phi}_1 \boldsymbol{s}_{1m}, \boldsymbol{\Phi}_2 \boldsymbol{s}_{2m}, \dots \boldsymbol{\Phi}_K \boldsymbol{s}_{Km})$, $\boldsymbol{A}$ the $K \times K$ diagonal matrix of received amplitudes, $\boldsymbol{H}[m] = \boldsymbol{S}[m]\boldsymbol{A}$, and $\boldsymbol{b}[m]$ and $\boldsymbol{y}[m]$ the vectors of transmitted and received signals, respectively. Furthermore, we decompose the matrix $\boldsymbol{H}[m]$ into two matrices of size $rN \times K$, $\boldsymbol{H}_u[m]$ and $\boldsymbol{H}_d[m]$ such that $\boldsymbol{H}[m] = [\boldsymbol{H}_u^T[m], \boldsymbol{H}_d^T[m]]^T$. Then, the baseband discrete-time asynchronous system in matrix notation is given by

$$\boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{N}} \tag{5}$$

where $\boldsymbol{\mathcal{Y}} = [\dots, \boldsymbol{y}^T[m-1], \boldsymbol{y}^T[m], \boldsymbol{y}^T[m+1]\dots]^T$ and $\boldsymbol{\mathcal{B}} = [\dots, \boldsymbol{b}^T[m-1], \boldsymbol{b}^T[m], \boldsymbol{b}^T[m+1]\dots]^T$ are the infinite-length vectors of received and transmitted symbols respectively; $\boldsymbol{\mathcal{N}}$ is an infinite-length noise vector; and $\boldsymbol{\mathcal{H}}$ is a bi-diagonal block matrix with infinite block rows and block columns given by

$$\boldsymbol{\mathcal{H}} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \dots & \boldsymbol{0} & \boldsymbol{H}_d[m-1] & \boldsymbol{H}_u[m] & \boldsymbol{0} & \dots & \dots \\ \dots & \dots & \boldsymbol{0} & \boldsymbol{H}_d[m] & \boldsymbol{H}_u[m+1] & \boldsymbol{0} & \dots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6}$$

Additionally, $\boldsymbol{h}_{k,m}$ denotes the column of the matrix $\boldsymbol{\mathcal{H}}$ containing the k^{th} column of the matrix $\boldsymbol{H}[m]$.

III. Linear MMSE Detection

The linear MMSE detector $\boldsymbol{\mathcal{C}}_{k,m}$ generates a soft decision $\widehat{b}_k[m] = \boldsymbol{\mathcal{C}}_{k,m}^H \boldsymbol{\mathcal{Y}}$ of the transmitted symbol $b_k[m]$ based on the observation $\boldsymbol{\mathcal{Y}}$. It is given by

$$\boldsymbol{\mathcal{C}}_{k,m} = (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^H + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_{k,m}. \tag{7}$$

The $\mathrm{SINR}_{k,m}$ at its output of the linear MMSE detector is given by

$$\mathrm{SINR}_{k,m} = \boldsymbol{h}_{k,m}^H (\boldsymbol{\mathcal{H}}_{k,m}\boldsymbol{\mathcal{H}}_{k,m}^H + \sigma^2 \boldsymbol{I})^{-1} \boldsymbol{h}_{k,m} \tag{8}$$

where, $\boldsymbol{\mathcal{H}}_{k,m}$ is the matrix obtained from $\boldsymbol{\mathcal{H}}$ suppressing the column $\boldsymbol{h}_{k,m}$.

Deeper insight on the linear MMSE behaviour is obtained by analyzing the performance, as $K, N \to \infty$ with constant ratio β. In order to determine the asymptotic SINR at the output of a linear MMSE detector we focus on CDMA systems such that the time delays of the received signals τ_k, $k = 1, \dots, K$ are not greater than the chip delay T_c, i.e. $\tau_k \leq T_c$, $k = 1, \dots, K$. We refer to them as *symbol quasi-synchronous but chip asynchronous* CDMA systems. Then, $\boldsymbol{\Phi}_k = \begin{bmatrix} \boldsymbol{C}_{\phi,r}^T(\tau_k) & \boldsymbol{0}_N^T \end{bmatrix}^T$, being $\boldsymbol{0}_N$ an $N \times N$ zero matrix. The matrix $\boldsymbol{\mathcal{H}}$ in (6) reduce to a block diagonal matrix with blocks of dimensions $rN \times K$ and we can focus on the transmission in a single symbol interval. The virtual spreading matrix in the m^{th} symbol interval is given by

$$\overline{\boldsymbol{S}}[m] = [\boldsymbol{C}_{\phi,r}(\tau_1)\boldsymbol{s}_{1m}, \boldsymbol{C}_{\phi,r}(\tau_2)\boldsymbol{s}_{2m} \dots \boldsymbol{C}_{\phi,r}(\tau_K)\boldsymbol{s}_{Km}]$$

Let $\overline{\boldsymbol{H}}[m] = \overline{\boldsymbol{S}}[m]\boldsymbol{A}$ be the transfer matrix of the system at time instant m^{th}. Without ambiguity we can drop the index m in the following and denote it by $\overline{\boldsymbol{R}} = \overline{\boldsymbol{H}}^H \overline{\boldsymbol{H}}$.

The following theorem provides the limit SINR at the output of a linear MMSE detector for user k and the multiuser efficiency defined as

$$\eta_k = \frac{N_0}{|a_{kk}|^2} \mathrm{SINR}_k. \tag{9}$$

Theorem 1 *Let $\boldsymbol{A} \in \mathbb{C}^{K \times K}$ be a diagonal matrix with k^{th} diagonal element a_{kk} and T_c a positive real. Given a function $\Phi(j2\pi f) : \mathbb{R} \to \mathbb{C}$, let $\phi(x, \tau)$ be as in (4). Given $\{\widetilde{\tau}_1, \widetilde{\tau}_2 \dots \widetilde{\tau}_K\}$ a set of reals in $[0, T_c]$ and an integer positive r, $\boldsymbol{C}_{\phi,r}(\widetilde{\tau}_k)$, $k = 1, \dots, K$, are K r-block-wise circulant matrices of order N defined in (3). Let $\overline{\boldsymbol{H}} = \overline{\boldsymbol{S}}\boldsymbol{A}$ with $\overline{\boldsymbol{S}} = (\boldsymbol{C}_{\phi,r}(\widetilde{\tau}_1)\boldsymbol{s}_1, \boldsymbol{C}_{\phi,r}(\widetilde{\tau}_2)\boldsymbol{s}_2, \dots, \boldsymbol{C}_{\phi,r}(\widetilde{\tau}_K)\boldsymbol{s}_K)$ and $\boldsymbol{s}_k$ N-dimensional column vectors.*

We assume that the function $\Phi(j2\pi f)$ is bounded in absolute value, bandlimited with bandwidth B and $r \geq 2BT_c$. The vectors $\boldsymbol{s}_k$ are independent with i.i.d. circulant symmetric Gaussian elements $s_{nk} \in \mathbb{C}$. The elements a_{kk} of the matrix $\boldsymbol{A}$ are uniformly bounded for any K. Furthermore, one of the following sets of conditions is satisfied.

Set of conditions A: *The sequence of the empirical joint distributions $F_{|\boldsymbol{A}|^2,T}^{(K)}(\lambda, \tau) = \frac{1}{K}\sum_{k=1}^{K} 1(\lambda - |a_{kk}|^2) 1(\tau - \widetilde{\tau}_k)$ converges almost surely, as $K \to \infty$, to a non-random distribution function $F_{|\boldsymbol{A}|^2,T}(\lambda, \tau)$ with λ and τ statistically independent and τ uniformly distributed in $[0, T_c]$.*

Set of conditions B: *The bandwidth B satisfies the constraint $B \leq \frac{1}{2T_c}$ and the sequence of the empirical distributions $F_{|\boldsymbol{A}|^2}(\lambda) = \frac{1}{K}\sum_{k=1}^{K} 1(\lambda - |a_{kk}|^2)$ converges in law almost surely to a deterministic distribution function.*

Then, given the power spectral density of the white noise N_0 the spectral efficiency of the linear MMSE detector for a CDMA system with transfer matrix $\overline{\boldsymbol{H}}$ converges in probability as $K, N \to \infty$ with $\frac{K}{N} \to \beta$ and r fixed to the deterministic value

$$\lim_{K=\beta N \to \infty} \eta_k = \eta(N_0) = \int_{-BT_c}^{BT_c} \eta(x, N_0)\, \mathrm{d}x \tag{10}$$

where the multiuser efficiency spectrum $\eta(x, N_0)$ is the unique solution to the fixed point equation

$$\frac{1}{\eta(x, N_0)} = \frac{T_c}{|\Phi\left(j2\pi \frac{x}{T_c}\right)|^2} + \beta \int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{N_0 + \lambda \int_{-BT_c}^{BT_c} \eta(x, N_0) \mathrm{d}x} \tag{11}$$

which is positive for $|x| \leq BT_c$.

Given $|a_{kk}|^2$, the received power of user k, the SINR of user k at the output of a linear MMSE detector converges in probability to

$$\lim_{K=\beta N \to \infty} \mathrm{SINR}_k = \frac{|a_{kk}|^2}{N_0}\eta(N_0) = \int_{-BT_c}^{BT_c} \mathrm{SINR}_k(x, N_0)\mathrm{d}x \tag{12}$$

where $\mathrm{SINR}_k(x, N_0) = \frac{|a_{kk}|^2}{N_0}\eta(x, N_0)$ is the spectrum of the SINR of user k.

Interestingly, (10), (11), and (12) admit the following interpretation. Let $P(x, \lambda) = \lambda \frac{r}{T_c^2} \left|\Phi\left(j2\pi \frac{x}{T_c}\right)\right|^2$ be the power density spectrum of the sampled received signal for a user having received power λ. Then, the limit SINR in Theorem 1 can be expressed as

$$\mathrm{SINR}_k(x) = \frac{P(x, |a_{kk}|^2)}{\sigma^2 + \beta \mathrm{E}_{|\boldsymbol{A}|^2}\{I(P(x,|a_{kk}|^2), P(x,\lambda), \mathrm{SINR}_k)\}}$$

with the effective interference density spectrum

$$I(P(x,|a_{kk}|^2), P(x,\lambda), \alpha_k) = \frac{P(x,|a_{kk}|^2)P(x,\lambda)}{P(x,|a_{kk}|^2) + P(x,\lambda)\mathrm{SINR}_k}.$$

Heuristically, this means that for large systems the SINR spectrum is deterministic and given by

$$\mathrm{SINR}_k(x) \approx \frac{P(x,|a_{kk}|^2)}{\sigma^2 + \frac{1}{N}\sum_{\substack{j=1\\ j\neq k}}^{K} I(P(x,|a_{kk}|^2), P(x,|a_{jj}|^2), \mathrm{SINR}_k)}.$$

Then, the interference at the frequency x can be decoupled into a sum of the background noise and an interference term from each of the users at the same frequency x. The total interference at frequency x depends only on the received power of the user of interest at the frequency x, the received power of the interfering user at the same frequency, and the attained SINR α_k. Therefore, in asynchronous systems we have a decoupling of the effects of interferers as in synchronous systems [2] and an additional decoupling in frequency.

From the set of conditions B, since no assumption is made on the time delay distribution, the large system performance is independent of the set of time delays for $B \leq \frac{1}{2T_c}$ and synchronous and asynchronous systems have the same performance. For $B > \frac{1}{2T_c}$ the equivalence between synchronous and asynchronous systems does not hold and the large system performance does depend on the time delay distribution. A general expression that holds for any time delay distribution with support $[0, T_c]$ is omitted here due to space restriction and can be found in [9]. For general time delays $\tau_k \in [0, T_s]$ we conjecture the equivalence in performance between the asynchronous systems and a chip asynchronous but symbol quasi synchronous system with time delays $\widetilde{\tau}_k = \tau_k - \left\lfloor \frac{\tau_k}{T_c} \right\rfloor T_c$. The rationale behind this conjecture is in [9].

The sinc functions with bandwidth $B = \frac{\gamma}{2T_c}$ have a particular theoretical interest. In the following we specialize Theorem 1 to this case.

Given a positive real γ,

$$\Phi(j2\pi f) = \begin{cases} \sqrt{\frac{T_c}{\gamma}} & \text{for } |f| \leq \frac{\gamma}{2T_c}, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

corresponds to a sinc waveform with bandwidth $B = \frac{\gamma}{2T_c}$ and unit energy. For large systems, the multiuser efficiency $\eta^{\mathrm{sinc}}(N_0)$ of a linear MMSE detector is the unique positive solution to the fixed point equation

$$\frac{1}{\eta^{(\mathrm{sinc})}(N_0)} = 1 + \frac{\beta}{\gamma}\int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{N_0 + \lambda \eta^{(\mathrm{sinc})}(N_0)}. \tag{14}$$

We recall that the multiuser efficiency of a linear MMSE detector for a synchronous CDMA system satisfies [2]

$$\frac{1}{\eta^{(\mathrm{syn})}(N_0)} = 1 + \beta \int \frac{\lambda \mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{N_0 + \lambda \eta^{(\mathrm{syn})}(N_0)}. \tag{15}$$

This result holds for synchronous CDMA systems using any chip pulse waveform with bandwidth $B \geq \frac{1}{2T_c}$ and satisfying the Nyquist criterion.

This result holds for synchronous CDMA systems using any chip pulse waveform with bandwidth $B \geq \frac{1}{2T_c}$ and satisfying the Nyquist criterion. Then, the comparison of (14) with (15) shows the interesting effect that an asynchronous CDMA system using a sinc function with bandwidth $B = \frac{\gamma}{2T_c}$ as chip pulse waveform performs as well as a synchronous CDMA system with system load $\beta' = \frac{\beta}{\gamma}$. This implies the possibility to trade the bandwidth of the chip pulse waveform against the spreading factor. In other words, we can trade the degrees of freedom in the frequency domain provided by the bandwidth of the chip pulse waveform against the degrees of freedom in the time domain provided by the spreading. This trading is typical of the asynchronous systems described above and does not extend to synchronous systems with the same waveform as apparent from (15).

IV. Capacity per Chip Constrained to a Chip Pulse Waveform

There exists a close relation between the total capacity of a CDMA system and the multiuser efficiency of a linear MMSE detector for the same system [3], [11]. The rationale behind this relation is a fundamental connection between mutual information and minimum mean-squared error in Gaussian channels [11]. In the following, we extend the results in Section III to get insight into the capacity per chip of an asynchronous CDMA system constrained to a chip pulse waveform, i.e. the capacity of a CDMA system for which the chip pulse waveforms for all the users and the chip intervals are identical to a given chip pulse waveform.

The total capacity per chip for large synchronous CDMA systems with random spreading in an additive white Gaussian noise channel is [1]

$$\mathcal{C}^{(\mathrm{syn})}(\beta, \mathrm{SNR}) = \beta \log_2\left(1 + \mathrm{SNR} - \frac{1}{4}\mathcal{F}(\mathrm{SNR}, \beta)\right) + \log_2\left(1 + \beta\mathrm{SNR} - \frac{1}{4}\mathcal{F}(\mathrm{SNR},\beta)\right) - \frac{\log_2 e}{4}\mathrm{SNR}\mathcal{F}(\mathrm{SNR},\beta)$$

being

$$\mathcal{F}(y,z) = \left(\sqrt{y(1+\sqrt{z})^2 + 1} - \sqrt{y(1-\sqrt{z})^2+1}\right)^2.$$

Consistently with the normalization adopted in the system model $\mathrm{SNR} = N_0^{-1}$.

The total capacity per chip of a synchronous CDMA system is equal to $\mathcal{C}^{(\mathrm{syn})}(\beta, \mathrm{SNR})$ for any square root Nyquist waveform.

The expression of the total capacity per chip for asynchronous CDMA systems constrained to a given chip pulse waveform $\phi(t)$ of bandwidth B can be obtained by making use of the results in Section III and the fundamental relation between mutual information and MMSE in Gaussian channels provided in [11]. In the following Corollary 1 we present the results for CDMA system satisfying the assumptions of Theorem 1.

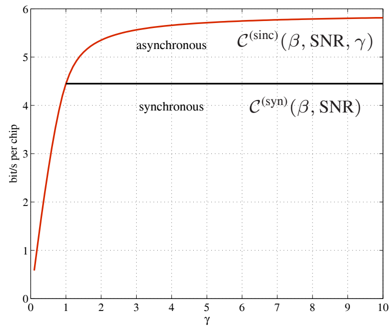

Fig. 1. Large system capacity per chip versus γ for $\beta = 1$ and $\frac{E_b}{N_0} = 10$ dB.

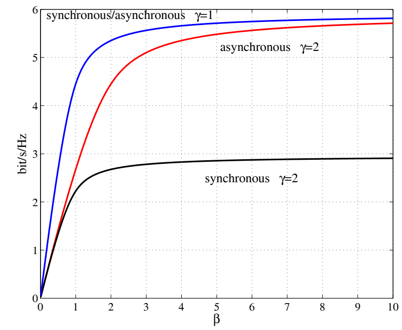

Fig. 2. Spectral efficiency versus β for $\frac{E_b}{N_0} = 10$ dB.

Corollary 1 *Let us adopt the same definitions and assumptions as in Theorem 1. Then, as $K, N \to \infty$ with $\frac{K}{N} \to \beta$, the total capacity per chip constrained to the chip pulse waveform $\phi(t)$ converge to a deterministic value*

$$\mathcal{C}^{(asyn)}\left(\beta, N_0^{-1}, \phi\right) = \frac{\beta}{\ln 2}\int_0^{N_0^{-1}} \mathrm{d}t \int_0^{+\infty} \frac{\lambda\eta(t)\mathrm{d}F_{|\boldsymbol{A}|^2}(\lambda)}{1+\lambda t\eta(t)}$$

where $\eta(t)$ is the multiuser efficiency of MMSE detectors in (10).

Let us consider again the case of chip pulse waveforms defined in (13), uniform distribution of the time delays, and additive white Gaussian channel without fading. Then, the total capacity per chip constrained to the chip pulse waveform (13) for large systems is

$$\mathcal{C}^{(\text{sinc})}(\beta, \text{SNR}, \gamma)\big|_{\text{SNR}=N_0^{-1}} = \frac{\beta}{\ln 2}\int_0^{{N_0}^{-1}} \frac{\eta^{(\text{sinc})}(t)\mathrm{d}t}{1+\lambda t\eta^{(\text{sinc})}(t)} \tag{16}$$

where $\eta^{(\text{sinc})}(t)$ is the solution to the fixed point equation (14). A similar equation holds also synchronous systems, when $\eta^{(\text{sinc})}(t)$ is replaced by the multiuser efficiency for synchronous system. As already noticed in Section III $\eta^{(\text{sinc})}(t)$ for an asynchronous system with load β equals $\eta^{(\text{syn})}(t)$ in (15) for a synchronous system with load $\beta' = \frac{\beta}{\gamma}$. Then, the following relation holds for the capacities

$$\mathcal{C}^{(\text{sinc})}(\beta, \text{SNR}, \gamma)\big|_{\text{SNR}=N_0^{-1}} = \gamma\, \mathcal{C}^{(\text{syn})}\left(\frac{\beta}{\gamma}, \text{SNR}\right)\bigg|_{\text{SNR}=N_0^{-1}}.$$

Similarly, it can be proven that the same relation holds also for flat fading channels. It is apparent from (16) that synchronous and asynchronous systems have the same capacity for $\gamma = 1$.

In order to compare different systems (with possibly different spreading gains and data rates) the total capacity per chip has to be given as a function of $\frac{E_b}{N_0}$, the level of energy per bit per noise level equal to [1] [3] $\frac{E_b}{N_0} = \frac{\beta\text{SNR}}{\mathcal{C}^{(*)}(\beta,\text{SNR},\cdot)}$.

In Figure 1 we compare the capacities per chip $\mathcal{C}^{(\text{sinc})}(\beta, \text{SNR}, \gamma)$ for asynchronous CDMA systems with $\mathcal{C}^{(\text{syn})}(\beta, \text{SNR})$ for synchronous CDMA systems. The capacities are plotted as functions of γ with $\frac{E_b}{N_0} = 10\,\text{dB}$ and $\beta = 1$. We see that asynchronous CDMA systems outperform synchronous systems and they compensate to some extent for the loss in spectral efficiency due to the increase in bandwidth of synchronous CDMA systems. In Figure 2 we compare the spectral efficiencies $\frac{\mathcal{C}^{(*)}(\beta,\text{SNR},\cdot)}{\gamma}$ of synchronous and asynchronous systems using the chip pulse waveform (13) for $\gamma = 1$ and $\gamma = 2$. The spectral efficiency is plotted as a function of the system load β for $\frac{E_b}{N_0} = 10$ dB. In contrast to the synchronous case, asymptotically for $\beta \to \infty$ the gap in spectral efficiency between synchronous/asynchronous systems with $\gamma = 1$ and asynchronous systems with $\gamma = 2$ vanishes.

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