Large System Design and Analysis of Protocols for Decode-Forward Relay Networks

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***Abstract*— In this work we consider a relay assisted CDMA network with a large number of sources and half duplex relays and a unique destination. We propose two relaying protocols called direct relaying (DR) and full relaying (FR). By dividing the relays in groups and adopting different forwarding delays for each group both protocols introduce diversity which depends on the number of groups and on the protocol. In DR mode, the relays forward only signals received directly from the sources. In FR mode, relays forward both signals received by the sources and the other relay groups by applying network coding at the physical layer. This implies a different level of diversity at the destination for the two schemes. Then, we propose an analytical framework for the analysis of the achievable rates in such a network, as the number of nodes and relays become asymptotically large.**

I. Introduction

In contrast to communications over one wireless point-to-point link, system design for a wireless network is vastly different due to that the number of users involved is large and that users can cooperate with each other to increase efficiency. For its complexity, there are various approaches to system design for wireless network communications, depending on different network abstraction models.

The simplest approach treats a wireless network as a collection of independent wireless point-to-point links connecting all the nodes [1]. Despite its simplicity, abstracting a wireless network into a set of independent links does not allow a sophisticated level of cooperation among users which may leads to potential increase in throughput. The simplest example is the use of relays (which can be acted by any network users) to assist communications between two users. A seminal groundbreaking work in relaying networks is by Cover and El Gamal [2] in which it considered a single relay scenario. The capacity of a degraded relay channel is obtained and is shown achievable by decode-and-forward scheme. For the general case, compress-and-forward (or quantize-and-forward) strategy was also proposed. A detailed discussion on various relaying strategies and their performances can be found in [3], [4]. While in the relay network, each node has a specific role – either a sender, a receiver or a relay. In cooperative communications, we take one step further by realizing that a node can assume all the three roles at the same time [5], [6], [7]. The fundamental question to be answered is what will the maximal throughput (i.e., communication rate per user) can be achieved subject to a specified total amount of network resources.

The performance analysis of such networks is very complex even when the number of nodes is small. It looks unaffordable for networks with an asymptotically large number of nodes, shortly referred to as *large networks* throughout this work. Nevertheless, some theories developed in physics resulted very effective in the analysis of large networks. Perculation theory, developed to analyze the flow of liquids through a porous body, enabled an insightful analysis of the information flows through extended or dense ad hoc networks (e.g. [17]). Random matrix theory, originally developed to investigate the levels of energies of electrons in atoms, resulted very effective in the analysis of large networks. In the seminal works [8], [9], random matrix theory is applied to investigate the spectral efficiency of large CDMA multiple access networks in case of additive white Gaussian noise and flat fading channels, respectively. Many works steamed from [8], [9]. As an example, random matrix theory is applied to analyze complex multicell CDMA networks, in [10].

In this paper, we aim at studying how relaying can increase network throughput when the number of senders and relaying nodes are large. For simplicity, we consider only the special case where communications are “unidirectional” – from senders to a common destination (e.g., a base station in a cellular network). Instead of deriving the ultimate capacity per user, we focus on simple strategies and evaluate their performance. Specifically, we consider two relaying strategies - the Direct Relaying (DR) strategy and the Full Relaying (FR) strategy. In the direct relaying strategies, relays decode and retransmit codewords if received directly from the sources. In contrast, in the full relaying strategies, relays forward also codewords received from other relays. We assume that all nodes share the same channel or band and the sources transmit continuously. The relays are half duplex and transmit only half of the time alternating the listening to the channel with the subsequent transmission of a codeword through a common channel. The relays are divided in $2L$ groups and each group forward codewords with a different delay compared to the original transmission of the sources. Thanks to these different forwarding delays the destination receives $L+1$ and $2L+1$ independent replicas of the same codeword for DR and FR mode, respectively. In the FR mode, the simultaneous retransmissions of codewords received directly from sources or decoded thanks to the forwarding of other groups of relays is performed by combining of the spread CDMA signals at the transmitting relays. This technique is also referred to as network coding at the physical layer (e.g. [13]) and have been proposed in [11], [12], [13].

By using random matrix theory, we provide a framework for the performance analysis of the system in terms of the achievable rate per user subject to (1) the system is large, i.e., both the number of source nodes N_s and the number of relays N_r are asymptotically large, (2) all sources and relays do not have complete channel state information, and (3) the spreading factor N grows with N_s and N_r such that $\frac{N_s}{N} \to \beta < \infty$ and $\frac{N_r}{N_s} \to \gamma < \infty$. This theoretical framework provides the achievable rates for the networks adopting the DR and FR schemes as a function of the noise variance, β, γ, and the limit distributions of the received powers at the relays and at the destination. We provide for each node an achievable rate constrained to the assumption of attainability of such a rate in all the other nodes. Then, the achievable rate per user in the network is the minimum among the above mentioned constrained rates achievable at the relays and the destination.

Additionally, we present a network model as case study to apply the proposed analytical framework.

II. System Model

Consider a synchronous mesh network where a large number of source nodes need to transmit to an access point and are supported by half duplex relay nodes in their transmission. Both source and relay nodes transmit synchronously using code division multiple access (CDMA) scheme with random spreading as multiple access protocol. Each relay decodes a subset of the received information and retransmits them using CDMA with a delay multiple of a codeword duration. Throughout this work we assume that the channels among all nodes (source-relay nodes, source-destination nodes and relay-destination) are flat fading and the signal at each receiving node (relays or destination) is impaired by white additive Gaussian noise.

We assume that there are N_s sources, namely $\mathcal{S}_k$, $k = 1, \ldots N_s$ transmitting to a destination node $\mathcal{D}$. In addition to these sources, there are $2L$ groups of N_r half-duplex relays, namely $\mathcal{R}_{ij}$ for $i = 1, \ldots, 2L$ and $j = 1, \ldots, N_r$. Each source or relay is equipped with a single antenna.

We use t as a time slot index. Source nodes transmit for all time slot t while a relay $\mathcal{R}_{ij}$ will transmit at time slot t only when $t - i$ is even. The codeword transmitted by $\mathcal{S}_k$ at t is denoted as $c_k(t)$. The m^{th} symbol in $c_k(t)$ is denoted as $c_{km}(t)$. The transmitted symbols $c_{km}(t)$ are assumed to be zero mean and unit variance random variables independent and identically distributed (i.i.d.) for all k, m and t. Each symbol $c_{km}(t)$ will further be spreaded into a chip-level sequence. Specifically, the transmitted chip level sequence $\boldsymbol{u}_{km}(t)$ for the symbol $c_{km}(t)$ by $\mathcal{S}_k$ is

$$\boldsymbol{u}_{km}(t) = \sqrt{P_k}\boldsymbol{s}_k(t)c_{km}(t) \tag{1}$$

where P_k is the transmitted power by user k and $\boldsymbol{s}_k(t)$ is a $N \times 1$ spreading chip sequence vector used by $\mathcal{S}_k$ at time t.

In this paper, we will consider three protocols - (i) the baseline model, (ii) the direct relaying (DR) model and (iii) the full relaying (FR) model. The differences among the three schemes are mainly on how relays assist transmission from the sources.

In the baseline model, relays did not transmit at all. In the DR protocol, relays decode and retransmit only codewords received directly from the sources, while in the FR protocol, relays also decode and forward codewords received from other relays. To precisely describe the three protocols, we first need to make a few definitions.

Let $\mathcal{U}_{ij}$ be the set of source codewords that the relay $\mathcal{R}_{ij}$ will decode and retransmit. We assume that $\mathcal{U}_{ij}$ is determined on the fly. Further details on how $\mathcal{U}_{ij}$ is determined will be given by the end of the section.

Recall that a relay $\mathcal{R}_{ij}$ can listen to source transmission while it is not transmitting (i.e., when $t - i$ is odd). Suppose that i is odd. Then the relay $\mathcal{R}_{ij}$ listens when t is even. In particular, it can decode $c_k(t')$ when t' is even. Similarly, when i is even, the relay $\mathcal{R}_{ij}$ can listen to the source transmission of $c_k(t')$ when t' is odd.

In the DR protocol, the relay will only retransmit codewords that it hears from the source directly. Specifically, for a relay $\mathcal{R}_{ij}$ where $t-i$ is even, it will retransmit the codeword $c_k(t-i-\delta(i))$ using a spreading chip sequence $\boldsymbol{s}^{(0)}_{ik}(t)$ where $\delta(i) = 1$ if i is even and equals zero otherwise. Note that the spreading chip sequence used by $\mathcal{R}_{ij}$ depends only on i. Therefore, the chip-level sequence $\boldsymbol{u}^{(0)}_{ijm}(t)$ transmitted by $\mathcal{R}_{ij}$ for the m^{th} symbol is

$$\boldsymbol{u}^{(0)}_{ijm}(t) = \sum_{k \in \mathcal{U}_{ij}} \sqrt{P^{(0)}_{ijk}}\boldsymbol{s}^{(0)}_{ik}(t)c_{km}(t - i - \delta(i)) \tag{2}$$

where $P^{(0)}_{ijk}$ is the power used by $\mathcal{R}_{ij}$ to retransmit codewords received from $\mathcal{S}_k$.

On the other hand in the FR model, a relay $\mathcal{R}_{ij}$ will not only retransmit $c_k(t - i - \delta(i))$ but also $c_k(t - i - \delta(i) - 1)$ which is obtained by listening to other relays' transmission. Let

$$\boldsymbol{u}^{(1)}_{ijm}(t) = \sum_{k \in \mathcal{U}_{ij}} \sqrt{P^{(1)}_{ijk}}\boldsymbol{s}^{(1)}_{ik}(t)c_{km}(t - i - \delta(i) - 1) \tag{3}$$

where $P^{(1)}_{ijk}$ is the power used by $\mathcal{R}_{ij}$ to retransmit the codeword $c_k(t - i - \delta(i) - 1)$ and $\boldsymbol{s}^{(1)}_{ik}(t)$ is a spreading chip sequence used by a relay at group i for codewords from $\mathcal{S}_k$.

The chip-level sequence transmitted by the relay in the FR mode for the m^{th} symbol is

$$\boldsymbol{u}_{ijm}(t) = \boldsymbol{u}^{(0)}_{ijm}(t) + \boldsymbol{u}^{(1)}_{ijm}(t). \tag{4}$$

when $t - i$ is even and is equal to zero otherwise.

To simplify notations, we use two parameters p and q to identify the three schemes. In the baseline model (i.e., no cooperation from relays), we let $q = 0$. In the DR model, we define $q = 1$ and $p = 0$ while both p and q are 1 in the FR mode.

Using the convention, the signal transmitted by relays are as follows:

$$\boldsymbol{u}_{ijm}(t) = q\boldsymbol{u}^{(0)}_{ijm}(t) + qp\boldsymbol{u}^{(1)}_{ijm}(t) \tag{5}$$

The transmission at the relays is illustrated in Figure 1 for both DR and FR mode. There we focus on the transmission of a user k and the forwarding of its information by relays for $L=2$, i.e. four groups of relays in DR and FR mode. In particular, for each group we consider a relay $\mathcal{R}_{ij}$ such that $k \in \mathcal{U}_{ij}$ and we show the forwarded information from user k in each frame.

Let $\boldsymbol{r}_m(t)$ be the received chip-level sequence at the sink $\mathcal{D}$ for the m^{th} symbol at t. Then

$$\boldsymbol{r}_m(t) = \sum_k a_k \boldsymbol{u}_{km}(t) + \sum_{ij} a_{ij}\boldsymbol{u}_{ijm}(t) + \boldsymbol{w}_m(t). \quad (6)$$

where a_k and a_{ij} are respectively channel gains from $\mathcal{S}_k$ and $\mathcal{R}_{ij}$ to the sink, and $\boldsymbol{w}_m(t)$ is the Gaussian noise.

Similarly, let $\boldsymbol{r}_m^{uv}(t)$ be the received sequence for the m^{th} symbol by the relay $\mathcal{R}_{uv}$ at time t. Then,

$$\boldsymbol{r}_m^{uv}(t) = \sum_k a_k^{uv} \boldsymbol{u}_{km}(t) + \sum_{ij} a_{ij}^{uv}\boldsymbol{u}_{ijm}(t) + \boldsymbol{w}_m^{(uv)}(t), \quad (7)$$

where a_k^{uv} and a_{ij}^{uv} are respectively the channel gains from source $\mathcal{S}_k$ and the relay $\mathcal{R}_{ij}$ to $\mathcal{R}_{uv}$ and $\boldsymbol{w}_m^{(uv)}(t)$ is the Gaussian noise.

Let $\boldsymbol{S}_0(t) = (\boldsymbol{s}_1(t), \boldsymbol{s}_2(t). \dots \boldsymbol{s}_{N_s}(t))$ be the $N \times N_s$ source spreading matrix whose k-th column is $\boldsymbol{s}_k(t)$. Similarly, for $s=1,2$, let $\boldsymbol{S}_i^{(s)}(t)$ be $(\boldsymbol{s}_{i1}^{(s)}(t), \boldsymbol{s}_{i2}^{(s)}(t) \dots \boldsymbol{s}_{iN_s}^{(s)}(t))$ be $N \times N_s$ relay spreading matrix for group i whose k-th column is $\boldsymbol{s}_{ik}^{(s)}(t)$.

By rearranging terms, (6) can be rewritten as

$$\begin{aligned}\boldsymbol{r}_m(t) = &\sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} q\boldsymbol{S}_i^{(0)}(t)\boldsymbol{H}_i^{(0)}\mathbf{c}_{*m}(t-i-\delta(i))\\ &+ \sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} pq\boldsymbol{S}_i^{(1)}(t)\boldsymbol{H}_i^{(1)}\mathbf{c}_{*,m}(t-i-\delta(i)-1)\\ &+ \boldsymbol{S}_0(t)\boldsymbol{H}_0\mathbf{c}_{*m}(t) + \boldsymbol{w}_m(t). \end{aligned} \quad (8)$$

where $\mathbf{c}_{*m}(t) \triangleq [c_{1m}(t), \cdots, c_{N_s m}(t)]^\top$, and $\boldsymbol{H}_0, \boldsymbol{H}_i^{(0)}$ and $\boldsymbol{H}_i^{(1)}$ are $N_s \times N_s$ diagonal matrices whose respective diagonal entries h_{0k}, $h_{ik}^{(0)}$ and $h_{ik}^{(1)}$ are defined by

$$h_{0k} = \sqrt{P_k} a_k \quad (9)$$

$$h_{ik}^{(s)} = \sum_{\substack{j=1\\ \mathcal{S}_k \in \mathcal{U}_{ij}}}^{N_r} \sqrt{P_{ijk}^{(s)}} a_{ij}. \quad (10)$$

Similarly, (7) can be rewritten as

$$\begin{aligned}\boldsymbol{r}_m^{uv}(t) = q &\sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} \boldsymbol{S}_i^{(0)}(t)\boldsymbol{H}_i^{(0,uv)}\mathbf{c}_{*m}(t-i-\delta(i))\\ &+ pq \sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} \boldsymbol{S}_i^{(1)}(t)\boldsymbol{H}_i^{(1,uv)}\mathbf{c}_{*,m}(t-i-\delta(i)-1)\\ &+ \boldsymbol{S}_0(t)\boldsymbol{H}_0^{uv}\mathbf{c}_{*m}(t) + \boldsymbol{w}_m^{(uv)}(t) \end{aligned} \quad (11)$$

where $\boldsymbol{H}_0^{uv}, \boldsymbol{H}_i^{(0,uv)}$ and $\boldsymbol{H}_i^{(1,uv)}$ are $N_s \times N_s$ diagonal matrices whose respective diagonal entries h_{0k}^{uv}, $h_{ik}^{(0,uv)}$ and $h_{ik}^{(1,uv)}$ are defined by

$$h_{0k}^{(uv)} = \sqrt{P_k} a_k^{uv} \quad (12)$$

$$h_{ik}^{(s,uv)} = \sum_{\substack{j=1\\ \mathcal{S}_k \in \mathcal{U}_{ij}}}^{N_r} \sqrt{P_{ijk}^{(s)}} a_{ij}^{uv}. \quad (13)$$

III. Large System Analysis

In this section we investigate the network performance as the system size grows large. More specifically, we assume that the spreading factor N, the number of users and relays N_s and N_r tend to infinity with asymptotic constant ratios, i.e. $\frac{N_s}{N} \to \beta$ and $\frac{N_r}{N_s} \to \gamma$. For this asymptotic analysis we assume that the transmitted symbols $c_{km}(t)$ are Gaussian i.i.d. distributed with zero mean and unit variance. The spreading sequences are independent over all sources, relays, and codewords with zero mean elements of variance $1/N$.

The achievable rate for the baseline system model in asymptotic conditions is in [9]. In order to derive the achievable rates for the DR and FR systems we use the results in [15] and [16] to derive the asymptotic SINR at the output of the MMSE detector. The fundamental results on the relation between MMSE and mutual information in a Gaussian vector channel provided in [14] are then applied to derive an achievable sum rate at the relays and the destination.

At the relay $\mathcal{R}_{uv}$, only codewords transmitted by the subset of sources $\mathcal{U}_{uv}$, with cardinality $|\mathcal{U}_{uv}| = \varphi N_s$, is decoded. We assume that the receiver ignores the interference structure due to codewords which are not decoded and considers it as an additional component of the additive Gaussian noise. This is equivalent to consider the following system

$$\begin{aligned}\overline{\boldsymbol{r}}_m^{uv}(t) = &\sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} pq\overline{\boldsymbol{S}}_i^{(0)}(t)\overline{\boldsymbol{H}}_i^{(0,uv)}\overline{\boldsymbol{c}}_{*m}(t-i-\delta(i))\\ &+ \sum_{\substack{i=1\\ i+t \text{ even}}}^{2L} pq\overline{\boldsymbol{S}}_i^{(1)}(t)\overline{\boldsymbol{H}}_i^{(1,uv)}\overline{\boldsymbol{c}}_{*m}(t-i-\delta(i))\\ &+ \overline{\boldsymbol{S}}_0(t)\overline{\boldsymbol{H}}_0^{uv}\overline{\boldsymbol{c}}_{*m}(t) + \overline{\boldsymbol{w}}_m^{(uv)}(t) \end{aligned} \quad (14)$$

where $\overline{\boldsymbol{S}}_i^{(0)}(t)$, $\overline{\boldsymbol{H}}_i^{(0,uv)}$ and $\overline{\boldsymbol{c}}_{*m}$ are obtained by suppressing columns, rows and columns and elements from $\boldsymbol{S}_i^{(0)}(t)$, $\boldsymbol{H}_i^{(0,uv)}$, and $\boldsymbol{c}_{*m}$, respectively. The vector $\overline{\boldsymbol{w}}_m^{(uv)}(t)$ is the white Gaussian noise with variance $\overline{\sigma}^2$ equal to the sum of the interference variance and the noise variance.

The following theorem provides the SINR for the estimate of a symbol $c_{km}(\ell)$ at the output of a MMSE filter as N, N_s, $N_r \to +\infty$ with constant ratios. We assume that the relay has an observation window unlimited in the past and limited to the frame $t \leq \ell$.

Theorem 1 *Let $\mathcal{R}_{uv}$ be a relay and u be odd. Let the received powers at the destination $|h_{0k}|^2$, and $|h_{ik}^{(s)}|^2$, for $i = 1, \dots 2L$, $s = 0, 1$ converge to deterministic marginal*

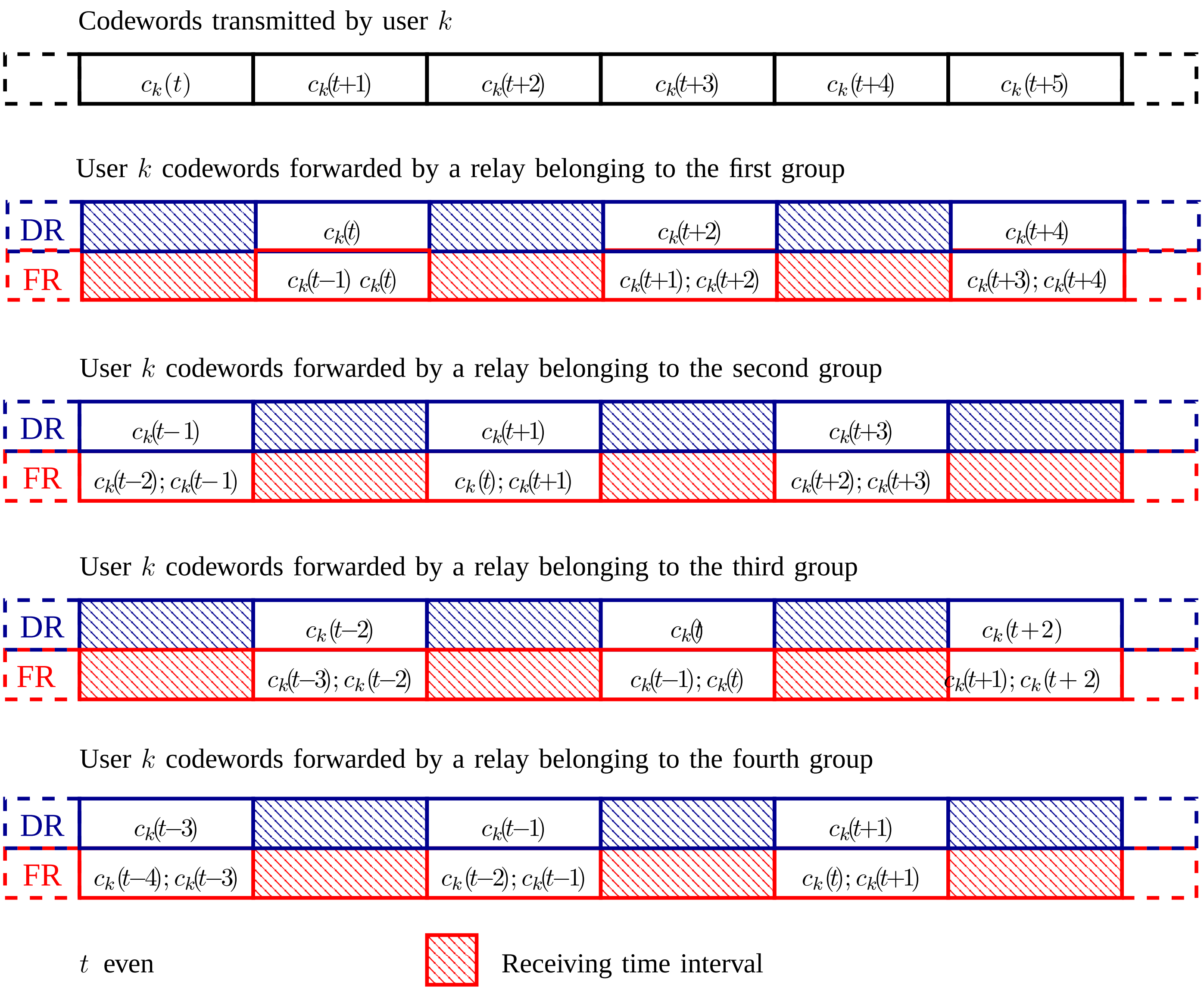

Fig. 1. Transmitted information at the relays in the DR and FR modes for $L = 2$ and t even.

and joint distribution functions $F_0(\lambda_0), F_i^{(s)}(\lambda_i^{(s)})$, and $F(\boldsymbol{\lambda})$ respectively, when $|\mathcal{U}_{uv}| = \varphi N_s \rightarrow \infty$. Then, given the received powers $|h_{0k}^{(uv)}|^2, |h_{2i,k}^{(s,uv)}|^2, i = 1, \ldots L$ and $s = 0, 1$ for $c_{km}(\ell)$, and the variance of the white noise $\overline{\sigma}^2$, the SINR of $c_{km}(\ell)$ at the output of a MMSE filter for a system in (14) and $t \geq \ell$ converges in probability as $N, N_s, N_r \rightarrow \infty$ with $\frac{N_s}{N} \rightarrow \beta$ to a deterministic limit

$$\lim_{\substack{N_s, N, N_r \rightarrow \infty \\ \frac{N_s}{N} \rightarrow \beta, \frac{N_r}{N_s} \rightarrow \gamma}} \mathrm{SINR}_k(\ell) = \begin{cases} p\sum_{i=1}^{L} |h_{2i,k}^{(1,uv)}|^2 \psi(\ell - 2i) \\ \quad + |h_k^{(uv)}|^2 \psi(\ell) \qquad \ell \text{ even}, \\ \\ p\sum_{i=1}^{L} |h_{2i,k}^{(0,uv)}|^2 \psi^{-1}(\ell - 2i + 1) \\ \qquad \ell \text{ odd}. \end{cases}$$

where $p = 0$ for DR systems and $p = 1$ for FR systems, and $\psi(u)$ is obtained as positive solution of the infinite system of fixed point equations[1]

$$\psi^{-1}(q) = \overline{\sigma}^2 + \varphi\beta \int \frac{\lambda_0^{(uv)} \mathrm{d}\, F(\boldsymbol{\lambda})}{1 + \lambda_0^{(uv)} \psi(q) + p\sum_{i=1}^{L} \lambda_{2i}^{(1,uv)} \psi(q + 2i)} + \varphi\beta p \sum_{i=1}^{L} (A_i^{(0)} + A_i^{(1)}) \qquad (15)$$

where q is an integer in $-\infty < q \leq t$,

$$A_i^{(0)} = \int \frac{\lambda_{2i}^{(0,uv)} \mathrm{d}\, F(\boldsymbol{\lambda})}{1 + \sum_{j=1}^{L} \lambda_{2j}^{(0,uv)} \psi(q + 2(i - j))}$$

and

$$A_i^{(1)} = \int \frac{\lambda_{2i}^{(1,uv)} \mathrm{d}\, F(\boldsymbol{\lambda})}{1 + \lambda_0^{(uv)} \psi(q + 2i) + \sum_{j=1}^{L} \lambda_{2j}^{(1,uv)} \psi(q + 2(i - j))}.$$

[1]Note that for the DR protocol the infinite system of equations reduces (15) to a single fixed point equation as the multiple access channel.

By convention, $\psi(q) = 0$ for $q > t$.
Furthermore,

$$\lim_{\substack{\ell\to+\infty\\ N_s,N,N_r\to\infty\\ \frac{N_s}{N}\to\beta,\,\frac{N_r}{N_s}\to\gamma}} \mathrm{SINR}_k(-2\ell-1) = p\sum_{i=1}^{L}|h_{2i,k}^{(0,uv)}|^2\psi$$

and

$$\lim_{\substack{\ell\to+\infty\\ N_s,N,N_r\to\infty\\ \frac{N_s}{N}\to\beta,\,\frac{N_r}{N_s}\to\gamma}} \mathrm{SINR}_k(-2\ell) = \left(|h_{0k}^{(uv)}|^2 + p\sum_{i=1}^{L}|h_{2i,k}^{(1,uv)}|^2\right)\psi$$

where ψ is the unique nonnegative solution of the fixed point equation

$$\psi^{-1} = \overline{\sigma}^2 + \varphi\beta\int\frac{\lambda_0^{(uv)} + p\sum_{i=1}^{L}\lambda_{2i}^{(1,uv)}\mathrm{d}\,F(\boldsymbol{\lambda})}{1+\left(\lambda_0^{(uv)}+p\sum_{j=1}^{L}\lambda_{2j}^{(1,uv)}\right)\psi} + p\varphi\beta\int\frac{\sum_{i=1}^{L}\lambda_{2i}^{(0,uv)}\mathrm{d}\,F(\boldsymbol{\lambda})}{1+\sum_{i=1}^{L}\lambda_{2i}^{(0,uv)}\psi}. \tag{16}$$

For the system in (7) with finite number of user and finite observation window $(-n,t)$, the mutual information per chip can be obtained from the results on the vector channel in [14] as

$$I^{(uv)}(t) = \sum_{\ell=n}^{t}\sum_{k=1}^{\phi N_s}\int_0^{\overline{\sigma}^{-2}}\frac{\mathrm{SINR}_k^{uv}(\ell,\rho)\mathrm{d}\,\rho}{\rho(1+\mathrm{SINR}_k^{uv}(\ell,\rho))} \tag{17}$$

where $\mathrm{SINR}_k^{uv}(\ell,\rho)$ is the SINR of symbol $c_{k,\ell}$ at the output of the MMSE detector as a function of $\rho=\sigma^{-2}$, being σ^2 the noise variance in the system.

The mutual information per chip and per channel use is given by

$$\mathcal{I}_c^{(uv)}(t) = \frac{I^{(uv)}(t)}{(t+n+1)N} \tag{18}$$

and

$$\mathcal{I}_u^{(uv)}(t) = \begin{cases}\dfrac{2I^{(uv)}(t)}{\varphi N_s(t+n+1)} & \text{for DR,}\\ \dfrac{I^{(uv)}(t)}{\varphi N_s(t+n+1)} & \text{for FR}\end{cases} \tag{19}$$

respectively[2].

The following corollary provides the the mutual information in the asymptotic limit when the system size grow large and the observation window is unlimited in the past, i.e. $n\to+\infty$.

Corollary 1 *Assume that the conditions in Theorem 1 are satisfied. When the observation window $(-n,t)$ is unlimited in the left, i.e. $n\to+\infty$, and N,N_s,N_r tend to infinity with asymptotically constant ratio, i.e. $\frac{N_s}{N}\to\beta>0$, and $\frac{N_r}{N_s}\to\gamma>0$, then the mutual information per chip of the channel (7) is given by*

$$\begin{aligned}\mathfrak{I}_c^{(uv)}(p) &= \lim_{\substack{n,N,N_s,N_r\to+\infty\\ \frac{N_s}{N}\to\beta,\,\frac{N_r}{N_s}\to\gamma}}\mathcal{I}_c(t)\\ &= \frac{\varphi\beta}{2}\int_0^{\overline{\sigma}^{-2}}\frac{\mathrm{d}\rho}{\rho}\int\sum_{s=0}^{1}\frac{\mathrm{SINR}^{(s)}(\boldsymbol{\lambda},\rho)\mathrm{d}\,F(\boldsymbol{\lambda})}{1+\mathrm{SINR}^{(s)}(\boldsymbol{\lambda},\rho)}\end{aligned} \tag{20}$$

where $\mathrm{SINR}^{(0)}(\boldsymbol{\lambda},\rho) = p(\sum_{i=1}^{L}\lambda_{2i}^{(0,uv)})\psi(\rho)$, $\mathrm{SINR}^{(1)}(\boldsymbol{\lambda},\rho) = (\lambda_0 + p\sum_{i=1}^{L}\lambda_{2i}^{(1,uv)})\psi(\rho)$, and $\psi(\rho)$ is solution to the fixed point equation (16) for a noise variance ρ^{-1}, $p=0$ in DR mode and $p=1$ in FR mode. The asymptotic mutual information per channel use $\mathfrak{I}_u^{(uv)}$ follows from the relation

$$\mathfrak{I}_u^{(uv)} = \begin{cases}\dfrac{2}{\varphi\beta}\mathfrak{I}_c^{(uv)}(0) & \textit{for DR},\\ \dfrac{1}{\varphi\beta}\mathfrak{I}_c^{(uv)}(1) & \textit{for FR}.\end{cases} \tag{21}$$

Similar equations hold when the relay $\mathcal{R}_{uv}$ belongs to an even group.

Now we consider the performance of the whole system at the destination. In this case we can assume an unlimited observation window in both directions of the time axis. This simplify considerably the analysis of the system performance in terms of the SINR provided in the following theorem.

Theorem 2 *Consider the system in (6) with $t\in(-\infty,+\infty)$. Let the received powers at the destination $|h_{0k}|^2$, and $|h_{ik}^{(s)}|^2$, for $i=1,\dots 2L$, $s=0,1$ converge to deterministic marginal and joint distribution functions $F_0(\lambda_0), F_i^{(s)}(\lambda_i^{(s)})$, and $F(\boldsymbol{\lambda})$ respectively, when $N_s,N_r\to\infty$. Furthermore, let $\mu_d^{(s)} = \sum_{i=1}^{L}\lambda_{2i-d}^{s}$, $s,d=0,1$ be random variables with probability distribution functions $F_M(\mu_d^{(s)})$. Denote by $F_M(\boldsymbol{\mu})$ the joint distribution of the multivariate random variable $\boldsymbol{\mu} = (\lambda_0,\mu_0^{(0)},\mu_0^{(1)},\mu_1^{(0)},\mu_1^{(1)})$. Then, given the received powers $|h_{0k}|^2$, and $|h_{ik}^{(s)}|^2$ for user k and the variance σ^2 of the Gaussian white noise at the receiver, the SINR of $c_{km}(\ell)$ at the output of a MMSE filter converges to a deterministic limit as $N,N_s,N_r\to+\infty$ with $\frac{N_s}{N}\to\beta$ and $\frac{N_r}{N_s}\to\gamma$*

$$\lim_{\substack{N,N_s,N_r\to+\infty\\ \frac{N_s}{N}\to\beta,\,\frac{N_r}{N_s}\to\gamma}}\mathrm{SINR}_k(\ell) = \begin{cases}(|h_k|^2 + qp\sum_{i=1}^{L}|h_{2i,k}^{(1)}|^2)\psi\\ \quad +q\sum_{i=1}^{L}|h_{2i-1,k}^{(0)}|^2\theta & \ell \textit{ even},\\ \\ (|h_k|^2 + qp\sum_{i=1}^{L}|h_{2i-1,k}^{(1)}|^2)\theta\\ \quad +q\sum_{i=1}^{L}|h_{2i-1,k}^{(0)}|^2\psi & \ell \textit{ odd},\end{cases} \tag{22}$$

where ψ and θ are the unique positive solutions to the system

[2]The factor 2 in the expression of the capacity per chip takes into account the fact that the channel is used only half of the time. The factor 2 in the expression of the capacity per channel use in the DR case takes into account the fact that only half of the codewords are decoded.

of two fixed point equations

$$\begin{aligned}\psi^{-1} &= \sigma^2 + \beta \int \frac{(\lambda_0 + p\mu_0^{(1)})\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0+p\mu_0^{(1)})\psi + \mu_1^{(0)}\theta} \\ &\quad + \beta \int \frac{\mu_0^{(0)}\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0+p\mu_1^{(1)})\theta + \mu_0^{(0)}\psi} \\ \theta^{-1} &= \sigma^2 + \beta \int \frac{(\lambda_0 + p\mu_1^{(1)})\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0+p\mu_1^{(1)})\theta + \mu_0^{(0)}\psi} \\ &\quad + \beta \int \frac{\mu_1^{(0)}\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0+p\mu_0^{(1)})\psi + \mu_1^{(0)}\theta} \qquad (23)\end{aligned}$$

If

$$F_M(\mu_1^s) = F_M(\mu_0^s) \qquad s = 0,1 \qquad (24)$$

then the limit (22) reduces to

$$\lim_{\substack{N,N_s,N_r\to+\infty \\ \frac{N_s}{N}\to\beta,\frac{N_r}{N_s}\to\gamma}} \mathrm{SINR}_k(\ell) = \begin{cases} \psi\left(\sum_{i=1}^L q(p)|h_{2i,k}^{(1)}|^2 + |h_{2i-1,k}^{(0)}|^2) \right. \\ \left. \quad +|h_{0k}|^2\right) \qquad \ell \textit{ even,} \\ \\ \psi\left(\sum_{i=1}^L q(p)|h_{2i-1,k}^{(1)}|^2 + |h_{2i-1,k}^{(0)}|^2) \right. \\ \left. \quad +|h_{0k}|^2\right) \qquad \ell \textit{ odd,} \end{cases} \qquad (25)$$

with ψ solution of the fixed point equation

$$\psi^{-1} = \sigma^2 + 2\beta \int \frac{(\lambda_0 + \mu_d^{(0)} + p\mu_d^{(1)})\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0 + \mu_d^{(0)} + p\mu_d^{(1)})\psi} \qquad (26)$$

If condition (24) is satisfied, the system at the destination is equivalent to a multiple access channel with distribution of the received powers equal to the distribution of the sum of the received powers from the source and the different groups of relays.

As for the relays, the results in [14] are applied to derive the mutual information at the destination of this relay assisted network.

Corollary 2 *Assume that the conditions of Theorem 2 are satisfied. Then, the mutual information per chip of the channel (6) as $N, N_s, N_r \to \infty$ with $\frac{N_s}{N} \to \beta$ and $\frac{N_r}{N_s} \to \gamma$ is*

$$\begin{aligned}\mathfrak{I}_c(p) &= \frac{\beta}{2}\int_0^\rho \frac{1}{\rho}\int \frac{((\lambda_0 + qp\mu_0^{(1)})\psi(\rho) + q\mu_1^{(0)}\theta(\rho))\mathrm{d}F(\boldsymbol{\mu})}{1+((\lambda_0+qp\mu_0^{(1)})\psi(\rho) + q\mu_1^{(0)}\theta(\rho))} \\ &\quad + \frac{\beta}{2}\int_0^\rho \frac{1}{\rho}\int \frac{((\lambda_0 + qp\mu_1^{(1)})\theta(\rho) + q\mu_0^{(0)}\theta(\rho))\mathrm{d}F(\boldsymbol{\mu})}{1+((\lambda_0+qp\mu_1^{(1)})\theta(\rho) + q\mu_0^{(0)}\psi(\rho))} \qquad (27)\end{aligned}$$

with $\theta(\rho)$ and $\psi(\rho)$ solutions of the system (23) for a noise level $\sigma^2 = \rho^{-1}$, $q = 0$ for the multiple access channel and $q = 1$ otherwise. Finally, $p = 0$ or $p = 1$ in DR mode or FD mode respectively.

If condition (24) is satisfied the mutual information per chip is given by

$$\mathfrak{I}_c = \beta\int_0^\rho \frac{1}{\rho}\int \frac{(\lambda_0 + qp\mu_d^{(1)} + q\mu_d^{(0)})\psi(\rho)\mathrm{d}F(\boldsymbol{\mu})}{1+(\lambda_0 + qp\mu_d^{(1)} + q\mu_d^{(0)})\psi(\rho)} \qquad (28)$$

where $\psi(\rho)$ is the solution to the fixed point equation (26) for a noise level $\sigma^2 = \rho^{-1}$.

Finally, the mutual information per channel use is

$$\mathfrak{I}_u(p) = \beta^{-1}\mathfrak{I}_c(p)$$

both for DR and FR relaying schemes.

Then, the achievable rate per channel use of the network is given by the minimum between the corresponding achievable rate at the destination and at all the relays.

Acknowledgment

The authors thanks Ralf Müller for very useful and insightful discussions.

This was supported by the French and Australian Science and Technology (FAST) Programme.

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