Weighted Sum Rate Maximization of Correlated MISO Broadcast Channels under Linear Precoding: A Large System Analysis

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Abstract—In this paper, we study the iterative weighted sum rate (WSR) maximizing linear precoder algorithm proposed by Christensen et al. in large correlated MISO broadcast channels. We propose a novel approximation of the signal-to-interference plus noise ratio (SINR) at every iteration. Moreover, for identical distributed channels the proposed approximation of the WSR maximizing precoder is of closed form and has the well known regularized zero-forcing (RZF) structure. Simulation results show that under correlated channels, the novel SINR approximation is more accurate than previously proposed estimates based on the RZF precoder.

I. Introduction

We consider the multiple-input single-output (MISO) broadcast channel (BC) with linear precoding at the transmitter. In this case, the base station (BS) is endowed with M antennas, whereas the K users have single-antenna receivers. The precoding matrix that maximizes (local optimum) the weighted sum rate (WSR) is obtained from an iterative algorithm [1]. Due to its iterative nature, this optimal precoder does not lend itself to analytic optimization problems that might give insightful results. Simple precoder such as zero-forcing (ZF) or regularized ZF (RZF) [2] are much more attractive in this regard. Therefore, it is important to understand the connection between the optimal iterative precoder and the suboptimal closed-form precoders. It has been shown in [3] that the iterative optimal precoder has the RZF structure, only when the channel is symmetric. Furthermore, it has been claimed in [4] that for large dimensions $M, K \to \infty$ with finite ratio $\beta \triangleq M/K$ and independent and identical distributed (i.i.d.) channels, the optimal linear precoder has the RZF structure as well.

In this contribution, we carry out a large system analysis of the iterative precoder proposed in [1]. Hereby, we extend the work in [5], [6] by deriving a novel approximation of the signal-to-interference plus noise ratio (SINR) at every iteration for large M. For per-user channel correlation, this new approximation outperforms the one proposed in [5], [6] and is identical to the RZF approximation for identical distributed channels [4], [7], [9].

The analysis leads to the following results, (i) under per-user channel correlation, the SINR approximation is iterative as well and is more accurate than the estimate obtained by a simplified RZF approach and (ii) for common user channel correlation, the precoder is of closed-form and has the RZF structure as proposed in [4]

Simulations show, that the proposed SINR approximation offers a good performance and is close to optimal.

Notation: In the following, boldface lower-case and upper-case characters denote vectors and matrices, respectively. The operators $(\cdot)^{\mathsf{H}}$, $\mathrm{tr}(\cdot)$ and $E[\cdot]$ denote conjugate transpose, trace and expectation, respectively. The $N \times N$ identity matrix is denoted $\mathbf{I}_N$ and $\log(\cdot)$ is the natural logarithm.

II. System Model

Consider a transmitter endowed with M antennas communicating with K single-antenna receivers. Under narrow-band transmission, the received signal y_k, $k = 1, 2, \ldots, K$, reads

$$y_k = \mathbf{h}_k^{\mathsf{H}} \mathbf{x} + n_k, \tag{1}$$

where $\mathbf{h}_k^{\mathsf{H}} \in \mathbb{C}^{1 \times M}$ is the channel vector of user k, $\mathbf{x} \in \mathbb{C}^M$ is the transmit vector and the n_k are independent complex Gaussian noise terms with zero mean and variance σ^2.

We assume that $\mathbf{h}_k^{\mathsf{H}}$ is correlated as $E[\mathbf{h}_k \mathbf{h}_k^{\mathsf{H}}] = \mathbf{\Theta}_k$ and thus can be written in the form

$$\mathbf{h}_k = \sqrt{M} \mathbf{\Theta}_k^{1/2} \mathbf{z}_k, \tag{2}$$

where $\mathbf{z}_k$ has i.i.d. complex entries of zero mean and variance $1/M$ and the $\mathbf{\Theta}_k^{1/2}$ is any Hermitian square-root of $\mathbf{\Theta}_k$. The correlation matrix $\mathbf{\Theta}_k$ is nonnegative Hermitian and of uniformly bounded spectral norm w.r.t. to M.

Prior to transmission, the user symbols $s_k \sim \mathcal{NC}(0, 1)$ are linearly precoded and form the transmit signal

$$\mathbf{x} = \sum_{k=1}^{K} \mathbf{g}_k s_k, \tag{3}$$

where $\mathbf{g}_k \in \mathbb{C}^M$ is the precoding vector of user k. The transmitter is subject to an average power constraint

$$E\left[\mathbf{x}^{\mathsf{H}} \mathbf{x}\right] = \mathrm{tr}\, \mathbf{G}\mathbf{G}^{\mathsf{H}} \leq P, \tag{4}$$

where $\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_K] \in \mathbb{C}^{M \times K}$ is the precoding matrix and P is the total available transmit power.

Under optimal single-user decoding, the achievable rate R_k of user k is given by

$$R_k = \log\left(1 + \gamma_k\right), \tag{5}$$

where γ_k is the SINR of user k and reads

$$\gamma_k = \frac{|\mathbf{h}_k^{\mathsf{H}} \mathbf{g}_k|^2}{\sum_{i \neq k}^{K} \mathbf{h}_k^{\mathsf{H}} \mathbf{g}_i \mathbf{g}_i^{\mathsf{H}} \mathbf{h}_k + \sigma^2}. \tag{6}$$

The precoding vectors are chosen to maximize the weighted sum rate of all users and the optimization problem takes the form

$$\mathbf{G}^{\star} = \arg\min_{\mathbf{G}} \sum_{k=1}^{K} -u_k R_k \tag{7}$$
$$\text{s.t.} \quad \text{tr}\,\mathbf{G}\mathbf{G}^{\mathsf{H}} \leq P,$$

where $u_k \geq 0$ is the weight of user k. The optimization problem in (7) is hard to solve directly, since it is highly nonconvex in the precoding matrix $\mathbf{G}$. To solve the problem in (7), consider the virtual linear receive filters $a_k \in \mathbb{C}$. The error variance e_k after the linear receive filtering reads

$$e_k = E\left[(a_k y_k - s_k)(a_k y_k - s_k)^{\mathsf{H}}\right]. \tag{8}$$

By introducing additional weighting scalars w_k, the cost function (7) can be modified and an equivalent optimization problem can be formulated as [1]

$$\{\mathbf{G}^{\star}, a_k^{\star}, w_k^{\star}\} = \arg\min_{\mathbf{G}, a_k, w_k} \left\{ u_k \left[w_k e_k - \log(u_k^{-1} w_k) - 1\right] \right\} \tag{9}$$
$$\text{s.t.} \quad \text{tr}\,\mathbf{G}\mathbf{G}^{\mathsf{H}} \leq P.$$

Denote $\rho \triangleq P/\sigma^2$ the signal-to-noise ratio (SNR). From (9), the optimal filters can be derived easily and read

$$a_k^{\star} = \sigma \mathbf{g}_k^{\mathsf{H}} \mathbf{h}_k \left(\sigma^2 + \mathbf{h}_k^{\mathsf{H}} \mathbf{G}\mathbf{G}^{\mathsf{H}} \mathbf{h}_k\right)^{-1} \tag{10}$$
$$e_k^{\star} = (1 + \gamma_k)^{-1} \tag{11}$$
$$w_k^{\star} = u_k (e_k^{\star})^{-1} \tag{12}$$
$$\tilde{\mathbf{G}}^{\star} = \left(\mathbf{H}^{\mathsf{H}} \mathbf{D} \mathbf{H} + \frac{\text{tr}\,\mathbf{D}}{\rho} \mathbf{I}_M\right)^{-1} \mathbf{H}^{\mathsf{H}} \mathbf{A}^{\mathsf{H}} \mathbf{W}, \tag{13}$$

where $\mathbf{G}^{\star} = \xi \tilde{\mathbf{G}}^{\star}$ with $\xi = \sqrt{P/\text{tr}\,\tilde{\mathbf{G}}^{\star} \tilde{\mathbf{G}}^{\star\mathsf{H}}}$. Also we defined $\mathbf{W} \triangleq \text{diag}(w_1^{\star}, \ldots, w_K^{\star})$, $\mathbf{A} \triangleq \text{diag}(a_1^{\star}, \ldots, a_K^{\star})$, $\mathbf{D} \triangleq \mathbf{A}^{\mathsf{H}} \mathbf{W} \mathbf{A}$ and $\mathbf{H} \triangleq [\mathbf{h}_1, \ldots, \mathbf{h}_K]^{\mathsf{H}} \in \mathbb{C}^{K \times M}$ is the compound channel. For notational convenience, we drop the superscript $\star$ in the sequel.

In this paper, we base our analysis on the iterative algorithm called *WSRBF-WMMSE2* proposed in [1]. Here, the precoding matrix is initialized with the transmit matched filter (MF). Subsequently a_k and w_k are computed, which then constitute the new precoder $\mathbf{G}$. This process is repeated until convergence to a local optimum.

III. LARGE SYSTEM ANALYSIS

In this section we derive an approximation $\bar{\gamma}_k$ of the SINR γ_k in (6) such that $\gamma_k - \bar{\gamma}_k \xrightarrow{M \to \infty} 0$, almost surely. This approximation is independent of the channel realizations and is referred to as *deterministic equivalent*.

At every iteration of the WSRBF-WMMSE2 algorithm, we calculate $\bar{\gamma}_k$, a deterministic equivalent of the SINR γ_k. Since the algorithm is iterative, $\bar{\gamma}_k^{(j)}$ at iteration j depends in a non-trivial way on the channel $\mathbf{H}$, making it difficult if not impossible to calculate. Therefore, we study the following precoder $\bar{\mathbf{G}}^{(j)}$ at iteration j,

$$\bar{\mathbf{G}}^{(j)} = \xi^{(j)} \tilde{\mathbf{G}}^{(j)}, \tag{14}$$

where

$$\tilde{\mathbf{G}}^{(j)} = \left(\mathbf{H}^{\mathsf{H}} \bar{\mathbf{D}}^{(j)} \mathbf{H} + \frac{\text{tr}\,\bar{\mathbf{D}}^{(j)}}{\rho} \mathbf{I}_M\right)^{-1} \mathbf{H}^{\mathsf{H}} \bar{\mathbf{A}}^{\mathsf{H},(j)} \bar{\mathbf{W}}^{(j)}, \tag{15}$$

$$\xi^{(j)} = \sqrt{P/\text{tr}\,\tilde{\mathbf{G}}^{(j)} \tilde{\mathbf{G}}^{\mathsf{H},(j)}} \tag{16}$$

with $\bar{\mathbf{D}}^{(j)} = \text{diag}(\bar{d}_1^{(j)}, \ldots, \bar{d}_K^{(j)})$, $\bar{\mathbf{A}}^{\mathsf{H},(j)} = \text{diag}(\bar{a}_1^{(j)}, \ldots, \bar{a}_K^{(j)})$ and $\bar{\mathbf{W}}^{(j)} = \text{diag}(\bar{w}_1^{(j)}, \ldots, \bar{w}_K^{(j)})$ such that $d_k^{(j)} - \bar{d}_k^{(j)} \xrightarrow{M \to \infty} 0$, $a_k^{(j)} - \bar{a}_k^{(j)} \xrightarrow{M \to \infty} 0$ and $w_k^{(j)} - \bar{w}_k^{(j)} \xrightarrow{M \to \infty} 0$ almost surely, respectively. That is, at iteration j, we use the *channel independent* approximations $\bar{a}_k^{(j)}$ and $\bar{w}_k^{(j)}$ to compute the precoding matrix $\bar{\mathbf{G}}^{(j)}$.

A. Deterministic Equivalent of the SINR for the MF

For the large system analysis, it is unimportant how we initialize the filters $\mathbf{g}_k$, because after the first iteration, the structure of the precoder is given by (14). As in the original algorithm in [1], we choose the transmit MF, i.e., $\mathbf{g}_k^{(0)} = \frac{\xi}{M} \mathbf{h}_k$. A deterministic equivalent $\bar{\gamma}_k^{\text{MF}}$ of the SINR γ_k^{MF} under MF precoding given by (6), such that $\gamma_k^{\text{MF}} - \bar{\gamma}_k^{\text{MF}} \xrightarrow{M \to \infty} 0$, almost surely, is given in the following theorem. We require $\bar{\gamma}_k^{\text{MF}}$ to compute $\bar{\gamma}_k^{(1)}$ in Theorem 2.

Theorem 1: Let γ_k^{MF} be the SINR of user k under MF precoding, i.e., $\mathbf{G} = \frac{\xi}{M} \mathbf{H}^{\mathsf{H}}$. Then, $\gamma_k^{\text{MF}} - \bar{\gamma}_k^{\text{MF}} \xrightarrow{M \to \infty} 0$, almost surely, where

$$\bar{\gamma}_k^{\text{MF}} = \frac{1}{\frac{1}{\beta\rho} + \frac{1}{M^2} \sum_{i \neq k}^{K} \text{tr}\,\mathbf{\Theta}_k \mathbf{\Theta}_i}. \tag{17}$$

Proof: The normalization parameter is $\xi = \sqrt{\frac{P}{\frac{1}{M^2} \text{tr}\,\mathbf{H}^{\mathsf{H}} \mathbf{H}}}$ and thus with (2) and [8, Lemma 2.7] we have ξ such that $\xi - \bar{\xi} \xrightarrow{M \to \infty} 0$ almost surely as

$$\bar{\xi} = \sqrt{\frac{P}{\frac{1}{M^2} \sum_{k=1}^{K} \text{tr}\,\mathbf{\Theta}_k}} = \sqrt{\beta P}. \tag{18}$$

Denote $P_{S,k} \triangleq |\mathbf{g}_k^{\mathsf{H}} \mathbf{h}_k|^2$ the signal power of user k. Applying [8, Lemma 2.7] we have $\frac{1}{M} \mathbf{h}_k^{\mathsf{H}} \mathbf{h}_k - 1 \xrightarrow{M \to \infty} 0$ and hence

$$\bar{P}_{S,k} = \bar{\xi}^2 = \beta P. \tag{19}$$

The interference is $\frac{\xi^2}{M} \mathbf{z}_k^{\mathsf{H}} \mathbf{\Theta}_k^{1/2} \mathbf{H}_{[k]}^{\mathsf{H}} \mathbf{H}_{[k]} \mathbf{\Theta}_k^{1/2} \mathbf{z}_k$, where $\mathbf{H}_{[k]} = [\mathbf{h}_1, \ldots, \mathbf{h}_{k-1}, \mathbf{h}_{k+1}, \ldots, \mathbf{h}_K]^{\mathsf{H}}$. Now we apply again [8,

Lemma 2.7] since $\frac{1}{M}\Theta_k^{1/2}\mathbf{H}_{[k]}^{\mathsf{H}}\mathbf{H}_{[k]}\Theta_k^{1/2}$ has uniformly bounded spectral norm w.r.t. M almost surely, and obtain

$$\frac{1}{M}\mathbf{z}_k^{\mathsf{H}}\Theta_k^{1/2}\mathbf{H}_{[k]}^{\mathsf{H}}\mathbf{H}_{[k]}\Theta_k^{1/2}\mathbf{z}_k - \frac{1}{M^2}\sum_{i\neq k}^{K}\text{tr}\Theta_k\Theta_i \xrightarrow{M\to\infty} 0, \tag{20}$$

almost surely. Substituting the terms in (6) by their respective deterministic equivalents yields (17), which completes the proof. ∎

B. Deterministic Equivalent of the SINR of Proposed Precoder

For the precoder (14), a deterministic equivalent of the SINR is provided in the following theorem.

Theorem 2: Let γ_k be the SINR of user k with the precoder defined in (14). Then, a deterministic equivalent $\bar{\gamma}_k^{(j)}$ at iteration $j > 0$ and under MF initialization $\bar{\mathbf{G}}^{(0)} = \frac{\xi^{(0)}}{M}\mathbf{H}^{\mathsf{H}}$, such that

$$\gamma_k^{(j)} - \bar{\gamma}_k^{(j)} \xrightarrow{M\to\infty} 0, \tag{21}$$

almost surely, is given by

$$\bar{\gamma}_k^{(j)} = \frac{\bar{w}_k^{(j)}(\bar{m}_k^{(j)})^2}{\bar{\Upsilon}_k^{(j)} + \bar{d}_k^{(j)}\frac{\bar{\Psi}^{(j)}}{\rho}(1+\bar{m}_k^{(j)})^2}, \tag{22}$$

where

$$\bar{m}_k^{(j)} = \frac{1}{M}\text{tr}\bar{\Theta}_k^{(j)}\mathbf{V}^{(j)}, \tag{23}$$

$$\bar{\Psi}^{(j)} = \frac{1}{M}\sum_{i=1}^{K}\frac{\bar{w}_i^{(j)}e_i'^{,(j)}}{(1+e_i^{(j)})^2}, \tag{24}$$

$$\bar{\Upsilon}_k^{(j)} = \frac{1}{M}\sum_{i=1}^{K}\frac{\bar{w}_i^{(j)}e_{i,k}'^{,(j)}}{(1+e_i^{(j)})^2}, \tag{25}$$

with $\bar{\Theta}_k^{(j)} = \bar{d}_k^{(j)}\Theta_k$ and $\bar{a}_k^{(j)}$, $\bar{w}_k^{(j)}$ and $\bar{d}_k^{(j)}$ are given by

$$\bar{a}_k^{(j)} = \frac{\sigma}{\sqrt{\bar{P}_{S,k}^{(j-1)}}}\frac{\bar{\gamma}_k^{(j-1)}}{1+\bar{\gamma}_k^{(j-1)}} \tag{26}$$

$$\sqrt{\bar{P}_{S,k}^{(j-1)}} = \frac{1}{\bar{a}_k^{(j-1)}}\sqrt{\frac{P}{\bar{\Psi}^{(j-1)}}}\frac{\bar{m}_k^{(j-1)}}{1+\bar{m}_k^{(j-1)}} \tag{27}$$

$$\bar{w}_k^{(j)} = u_k\left(1+\bar{\gamma}_k^{(j-1)}\right) \tag{28}$$

$$\bar{d}_k^{(j)} = \bar{w}_k^{(j)}\bar{a}_k^{2,(j)}. \tag{29}$$

Denoting $\mathbf{V}^{(j)} \triangleq \left(\mathbf{F}^{(j)} + \bar{\alpha}^{(j)}\mathbf{I}_M\right)^{-1}$ with $\bar{\alpha}^{(j)} \triangleq \frac{\text{tr}\bar{\mathbf{D}}^{(j)}}{M\rho}$, three systems of K coupled equations have to be solved. First, the $e_1^{(j)},\ldots,e_K^{(j)}$ form the unique positive solutions of

$$e_i^{(j)} = \frac{1}{M}\text{tr}\bar{\Theta}_i^{(j)}\mathbf{V}^{(j)}, \tag{30}$$

$$\mathbf{F}^{(j)} = \frac{1}{M}\sum_{i=1}^{K}\frac{\bar{\Theta}_i^{(j)}}{1+e_i^{(j)}}. \tag{31}$$

Secondly, the $e_1'^{,(j)},\ldots,e_K'^{,(j)}$ form the unique positive solutions of

$$e_i'^{,(j)} = \frac{1}{M}\text{tr}\bar{\Theta}_i^{(j)}\mathbf{V}^{2,(j)}\left(\mathbf{F}'^{,(j)}+\mathbf{I}_M\right), \tag{32}$$

$$\mathbf{F}'^{,(j)} = \frac{1}{M}\sum_{i=1}^{K}\frac{\bar{\Theta}_i^{(j)}e_i'^{,(j)}}{(1+e_i^{(j)})^2}. \tag{33}$$

Finally, the $e_{1,k}'^{,(j)},\ldots,e_{K,k}'^{,(j)}$ are the unique positive solutions of

$$e_{i,k}'^{,(j)} = \frac{1}{M}\text{tr}\bar{\Theta}_i^{(j)}\mathbf{V}^{2,(j)}\left(\mathbf{F}_k'^{,(j)}+\bar{\Theta}_k^{(j)}\right), \tag{34}$$

$$\mathbf{F}_k'^{,(j)} = \frac{1}{M}\sum_{i=1}^{K}\frac{\bar{\Theta}_i^{(j)}e_{i,k}'^{,(j)}}{(1+e_i^{(j)})^2}. \tag{35}$$

For $j = 1$, $\bar{\gamma}_k^{(0)} = \bar{\gamma}_k^{\text{MF}}$, given by Theorem 1 and $\bar{P}_{S,k}^{(0)} = \beta P$, cf. (19).

Proof: For $j > 0$, define $\Gamma^{(j)} \triangleq \frac{1}{M}\mathbf{H}^{\mathsf{H}}\bar{\mathbf{D}}^{(j)}\mathbf{H} + \bar{\alpha}^{(j)}\mathbf{I}_M$, where $\bar{\alpha}^{(j)} \triangleq \frac{\text{tr}\bar{\mathbf{D}}^{(j)}}{M\rho}$. The precoder at the end of iteration j is given by

$$\bar{\mathbf{G}}^{(j)} = \frac{\xi^{(j)}}{M}(\Gamma^{(j)})^{-1}\mathbf{H}^{\mathsf{H}}\bar{\mathbf{A}}^{\mathsf{H},(j)}\bar{\mathbf{W}}^{(j)}, \tag{36}$$

where $\xi^{(j)}$ is

$$\xi^{(j)} = \sqrt{\frac{P}{\frac{1}{M^2}\text{tr}(\Gamma^{(j)})^{-2}\mathbf{H}^{\mathsf{H}}\bar{\mathbf{A}}^{\mathsf{H},(j)}\bar{\mathbf{W}}^{2,(j)}\bar{\mathbf{A}}^{(j)}\mathbf{H}}} \tag{37}$$

$$\triangleq \sqrt{\frac{P}{\Psi^{(j)}}}. \tag{38}$$

The derivation of the deterministic equivalents of the normalization term $\xi^{(j)}$, the signal power $|\bar{\mathbf{g}}_k^{\mathsf{H},(j)}\mathbf{h}_k|^2$ and the interference power $\sum_{i\neq k}^{K}\mathbf{h}_k^{\mathsf{H}}\bar{\mathbf{g}}_i^{(j)}\bar{\mathbf{g}}_i^{\mathsf{H},(j)}\mathbf{h}_k$ is made equivalent to [5], [6] by replacing $\bar{\Theta}_k^{(j)} \triangleq \bar{d}_k^{(j)}\Theta_k$ and the user powers by the MMSE weights, i.e., $p_k = \bar{w}_k^{(j)}$. This will be shown in the following.

a) Power normalization $\xi^{(j)}$: The term $\Psi^{(j)}$ can be written as

$$\Psi^{(j)} = \frac{1}{M^2}\text{tr}(\Gamma^{(j)})^{-2}\mathbf{H}^{\mathsf{H}}\bar{\mathbf{A}}^{\mathsf{H},(j)}\bar{\mathbf{W}}^{2,(j)}\bar{\mathbf{A}}^{(j)}\mathbf{H} \tag{39}$$

$$= \frac{1}{M}\sum_{k=1}^{K}\bar{w}_k^{(j)}\bar{d}_k^{(j)}\mathbf{z}_k^{\mathsf{H}}\Theta_k^{1/2}(\Gamma^{(j)})^{-2}\Theta_k^{1/2}\mathbf{z}_k \tag{40}$$

$$= \frac{1}{M}\sum_{k=1}^{K}\bar{w}_k^{(j)}\mathbf{z}_k^{\mathsf{H}}\bar{\Theta}_k^{1/2,(j)}(\Gamma^{(j)})^{-2}\bar{\Theta}_k^{1/2,(j)}\mathbf{z}_k. \tag{41}$$

Similar to [5], [6], a deterministic equivalent $\bar{\Psi}^{(j)}$ such that $\Psi^{(j)} - \bar{\Psi}^{(j)} \xrightarrow{M\to\infty} 0$, almost surely, is given by

$$\bar{\Psi}^{(j)} = \frac{1}{M}\sum_{k=1}^{K}\bar{w}_k^{(j)}\frac{\frac{1}{M}\text{tr}\bar{\Theta}_k^{(j)}(\Gamma^{(j)})^{-2}}{\left(1+\frac{1}{M}\text{tr}\bar{\Theta}_k^{(j)}(\Gamma^{(j)})^{-1}\right)^2} \tag{42}$$

$$= \frac{1}{M}\sum_{k=1}^{K}\bar{w}_k^{(j)}\frac{\bar{m}_k'^{,(j)}}{\left(1+\bar{m}_k^{(j)}\right)^2} = \frac{1}{M}\sum_{k=1}^{K}\frac{\bar{w}_k^{(j)}e_k'}{(1+e_k)^2}, \tag{43}$$

where we denote $\bar{m}_k^{(j)} \triangleq \frac{1}{M}\mathrm{tr}\bar{\Theta}_k^{(j)}(\Gamma^{(j)})^{-1}$ and $\bar{m}_k'^{,(j)}$ is the derivative w.r.t. z at $z = -\bar{\alpha}^{(j)}$.

b) Signal power $|\bar{\mathbf{g}}_k^{\mathsf{H},(j)}\mathbf{h}_k|^2$: The square-root of the signal power $P_{S,k}^{(j)} \triangleq |\bar{\mathbf{g}}_k^{\mathsf{H},(j)}\mathbf{h}_k|^2$ is

$$\sqrt{P_{S,k}^{(j)}} = \xi^{(j)}\bar{a}_k^{(j)}\bar{w}_k^{(j)}\mathbf{z}_k^{\mathsf{H}}\Theta_k^{1/2}(\Gamma^{(j)})^{-1}\Theta_k^{1/2}\mathbf{z}_k \quad (44)$$

$$= \frac{\xi^{(j)}}{\bar{a}_k^{(j)}}\mathbf{z}_k^{\mathsf{H}}\bar{\Theta}_k^{1/2,(j)}(\Gamma^{(j)})^{-1}\bar{\Theta}_k^{1/2,(j)}\mathbf{z}_k. \quad (45)$$

Again, following [5], [6], a deterministic equivalent $\sqrt{\bar{P}_{S,k}^{(j)}}$ of (44) such that $\sqrt{P_{S,k}^{(j)}} - \sqrt{\bar{P}_{S,k}^{(j)}} \xrightarrow{M\to\infty} 0$, almost surely, is given by

$$\sqrt{\bar{P}_{S,k}^{(j)}} = \frac{\bar{\xi}^{(j)}}{\bar{a}_k^{(j)}}\frac{\bar{m}_k^{(j)}}{1+\bar{m}_k^{(j)}}, \quad (46)$$

where $\bar{\xi}^{(j)} = \sqrt{P/\bar{\Psi}^{(j)}}$. From (10), we have $a_k^{(j)} = \frac{\sigma}{\sqrt{P_{S,k}^{(j-1)}}}\frac{\gamma_k^{(j-1)}}{1+\gamma_k^{(j-1)}}$ for which $\bar{a}_k^{(j)}$ such that $a_k^{(j)} - \bar{a}_k^{(j)} \xrightarrow{M\to\infty} 0$ almost surly is given by (26).

c) Interference power $\sum_{i\neq k}^K \mathbf{h}_k^{\mathsf{H}}\bar{\mathbf{g}}_i^{(j)}\bar{\mathbf{g}}_i^{\mathsf{H},(j)}\mathbf{h}_k$: The interference power can be written as

$$\sum_{i\neq k}^K \mathbf{h}_k^{\mathsf{H}}\bar{\mathbf{g}}_i^{(j)}\bar{\mathbf{g}}_i^{\mathsf{H},(j)}\mathbf{h}_k$$

$$= \frac{\xi^{2,(j)}}{M^2}\mathbf{h}_k^{\mathsf{H}}(\Gamma^{(j)})^{-1}\sum_{i\neq k}^K \bar{a}_i^{2,(j)}\bar{w}_i^{2,(j)}\mathbf{h}_i\mathbf{h}_i^{\mathsf{H}}(\Gamma^{(j)})^{-1}\mathbf{h}_k \quad (47)$$

$$= \frac{\xi^{2,(j)}}{\bar{d}_k^{(j)}}\mathbf{z}_k^{\mathsf{H}}\bar{\Theta}_k^{1/2,(j)}(\Gamma^{(j)})^{-1}$$

$$\times \sum_{i\neq k}^K \bar{w}_i^{(j)}\bar{\Theta}_i^{1/2,(j)}\mathbf{z}_i\mathbf{z}_i^{\mathsf{H}}\bar{\Theta}_i^{1/2,(j)}(\Gamma^{(j)})^{-1}\bar{\Theta}_k^{1/2,j}\mathbf{z}_k. \quad (48)$$

Replacing p_k by $\bar{w}_k^{(j)}$, the term in (48) is equivalent to the one in [5], [6]. Thus we have

$$\sum_{i\neq k}^K \mathbf{h}_k^{\mathsf{H}}\bar{\mathbf{g}}_i^{(j)}\bar{\mathbf{g}}_i^{\mathsf{H},(j)}\mathbf{h}_k - \frac{\bar{\xi}^{2,(j)}\bar{\Upsilon}_k^{(j)}}{\bar{d}_k^{(j)}(1+\bar{m}_k^{(j)})^2} \xrightarrow{M\to\infty} 0, \quad (49)$$

almost surely, where $\bar{\Upsilon}_k^{(j)}$ is given by (25), which completes the proof. ∎

C. Common Correlation $\Theta_k = \Theta$ and Sum Rate $\mathbf{U} = \mathbf{I}_K$

If the correlation of all user channels is identical and sum rate is considered, the precoder in (14) is of closed form and independent of iteration j.

Proposition 1: Let $\Theta_k = \Theta$ and $\mathbf{U} = \mathbf{I}_K$, the the precoder in (14) is given by

$$\bar{\mathbf{G}} = \xi\left(\mathbf{H}^{\mathsf{H}}\mathbf{H} + \frac{K}{\rho}\mathbf{I}_M\right)^{-1}\mathbf{H}^{\mathsf{H}}, \quad (50)$$

where $\xi = \sqrt{P/\mathrm{tr}\bar{\mathbf{G}}\bar{\mathbf{G}}^{\mathsf{H}}}$.

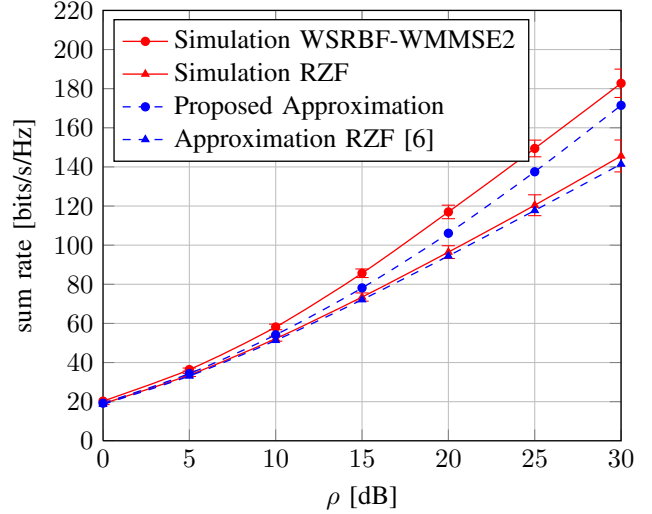
Fig. 1. Sum rate vs. SNR with $M = K = 30$, $\Theta_k \neq \mathbf{I}_M$, error bars indicating the standard deviation.

Proof: From Theorem 2, observe that both $\bar{a}^{(j)} \triangleq \bar{a}_k^{(j)}$ and $\bar{w}^{(j)} \triangleq \bar{w}_k^{(j)}$ are identical for all users and thus $\bar{\mathbf{D}}^{(j)} = \bar{a}^{2,(j)}\bar{w}^{(j)}$, Therefore, (14) takes the form

$$\tilde{\mathbf{G}}^{(j)} = \frac{\xi}{\bar{a}^{(j)}}\left(\mathbf{H}^{\mathsf{H}}\mathbf{H} + \frac{K}{\rho}\mathbf{I}_M\right)^{-1}\mathbf{H}^{\mathsf{H}} \quad (51)$$

$$\xi^2 = \frac{\bar{a}^{2,(j)}P}{\mathrm{tr}\left(\mathbf{H}^{\mathsf{H}}\mathbf{U}\mathbf{H} + \frac{K}{\rho}\mathbf{I}_M\right)^{-2}\mathbf{H}^{\mathsf{H}}\mathbf{H}}, \quad (52)$$

which completes the proof. ∎

Corollary 1: [5], [9] Let $\Theta_k = \Theta$ and $\mathbf{U} = \mathbf{I}_K$ then $\bar{\gamma}_k^{(j)} \triangleq \bar{\gamma}_k$ takes the form

$$\bar{\gamma} = \bar{m} = \frac{1}{M}\mathrm{tr}\Theta\left(\frac{\Theta/\beta}{1+\bar{m}} + \frac{1}{\beta\rho}\mathbf{I}_M\right)^{-1} \quad (53)$$

Corollary 2: [5], [6] Let $\Theta = \mathbf{I}_M$ and $\mathbf{U} = \mathbf{I}_K$, then $\bar{\gamma}_k^{(j)} \triangleq \bar{\gamma}_k$ is of closed form an reads

$$\bar{\gamma}_k^{(j)} \triangleq \bar{\gamma} = \frac{1}{2}\left[\rho(\beta-1) + \chi - 1\right], \quad (54)$$

where χ is given by

$$\chi = \sqrt{(\beta-1)^2\rho^2 + 2(1+\beta)\rho + 1}. \quad (55)$$

IV. NUMERICAL RESULTS

The Monte-Carlo (MC) simulations are carried out over independent Rayleigh block-fading channels $\mathbf{z}_k$. We use the WSRBF-WMMSE2 algorithm in [1] with 10 iterations and MF initialization and compare it to the large system approximation in Theorem 2. In all simulations, the approximated SINR in Theorem 2 at iteration $j = 3$ is computed and an approximated sum rate $\bar{R}_{\mathrm{sum}} = \sum_{k=1}^K \log(1 + \bar{\gamma}_k^{(3)})$ is obtained.

The channel correlation matrix is modeled as [10]

$$[\Theta_k]_{ij} = \frac{1}{\theta_{k,\max} - \theta_{k,\min}}\int_{\theta_{k,\min}}^{\theta_{k,\max}} e^{\mathbf{j}\frac{2\pi}{\lambda}d_{ij}\cos(\theta)}d\theta, \quad (56)$$

where $\mathbf{j} \triangleq \sqrt{-1}$, λ denotes the signal wavelength and d_{ij} is the distance between antenna i and j. As a simple example we choose the range of azimuth angle θ_k of user k as $\theta_{k,\min} = -\pi$ and $\theta_{k,\max} = \varphi_k - \pi$, where $\varphi_k = 2\pi k/K$ (uniform user distribution). The transmitter is endowed with a uniform linear array (ULA) of antennas. We assume that d_{ij} is independent of M so that the spectral norm of $\mathbf{\Theta}_k$ remains bounded as M grows large.

In Figure 1 we compare the optimal iterative linear precoders to the closed-form RZF precoder, in both MC simulations and their respective approximations under the correlation model (56) (indicated $\mathbf{\Theta}_k \neq \mathbf{I}_M$). It can be observed that (i) WSRBF-WMMSE2 is superior to RZF and (ii) the approximation for RZF [6] is more accurate than the proposed approximation for the WSRBF-WMMSE2. The first observation is expected, since the RZF is suboptimal. The second observation is explained by the fact that, for RZF it has been proved [6] that the approximation of the SINR is almost surely exact as $M \to \infty$. Still, the proposed approximation outperforms the RZF approximation and is not too far from optimal.

Moreover, other simulations suggest, that to be more precise, the approximation in Theorem 2 requires larger dimensions than the RZF equivalent.

Figure 2 compares the WSRBF-WMMSE2 precoder and its approximation for correlated channels ($\mathbf{\Theta}_k \neq \mathbf{I}_M$) and i.i.d. channels ($\mathbf{\Theta}_k = \mathbf{I}_M$). In [4], it has been claimed that for $\mathbf{\Theta}_k = \mathbf{I}_M$ the approximation of the SINR of RZF in Corollary 2 is almost surly optimal as $M \to \infty$. It can be observed that for i.i.d. channels the approximation in Corollary 2 is accurate for low SNR, but very imprecise at high SNR. This observation suggests that at least significantly higher dimensions are required to close the gap between approximation and simulation at high SNR. The reason is, that the regularization term in the inverse of (15) is going to zero for $\rho \to \infty$ and the maximum eigenvalue of the inverse is fluctuating significantly leading to large standard deviations in the simulations as well as to less accurate approximations. As Figure 2 suggests, this effect is diminished when the channel is correlated resulting in an increased accuracy of the approximation for high SNR.

V. CONCLUSION

This paper investigated the behavior of the optimal linear precoder in large correlated MISO broadcast channels. We derived a novel iterative approximation of the SINR and showed that it outperforms the closed-form RZF precoder. The novel approximation is independent of a particular channel realization and thus allows for less complex performance evaluation.

Fig. 2. WSRBF-WMMSE2, sum rate vs. SNR with $M = K = 30$, error bars indicating the standard deviation.

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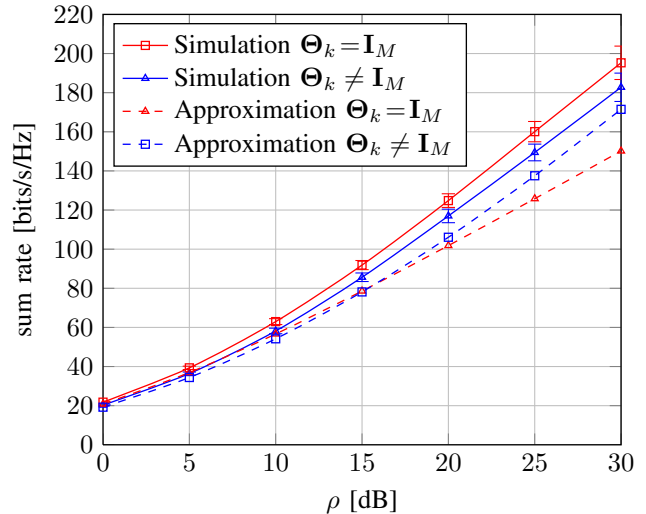
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