Spectral Efficiency of Extended Networks with Randomly Distributed Transmitters and Receivers

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***Abstract*—We consider an extended wireless network with transmit and receive nodes homogeneously distributed in $\mathbb{R}^2$. The channel state information (CSI) is known at the centralized processing unit at the receiver side but not at the transmitters which transmit at equal power. By extending the definition of Euclidean random matrices (ERM) and their spectral analysis to two independent sets of points, we determine an approximated expression of the average maximum achievable rate per unit area of the system.**

I. Introduction

Technologies as cloud radio access networks (C-RAN) and remote radio heads (RRH) are currently object of intensive studies and are expected to be strategic pillars of the future 5G wireless networks. These technologies will make technically feasible the concept of distributed antenna systems (DAS) with highly centralized processing points. DAS, introduced in [1], [2], are receiving large attention. Significant efforts have been already spent in the analysis of cellular DAS in downlink [3]–[6] assuming that the antennas are deployed at fix locations in a regular grid or as Poisson Point Processes with a certain intensity. This latter assumption opened the way to the application of stochastic geometry [7]. A detailed overview can be found in [6]. Multiuser DAS in uplink were studied in [8] and [9] using random matrix theory [10] and considering a given finite area. Due to strong underlying approximations, in both cases, the DAS sum capacity is approximated by the limiting capacity of a multiple input multiple output (MIMO) cellular system with full base stations cooperation, as pointed out in [11]. The good match between simulations and theoretical results appears when the transmit antennas are much more than receive antennas or when the average received signal to noise ratio (SNR) is low, because the system becomes insensitive to the receive antenna layout [11]. In [11] a single cell of given area with receive antennas homogeneously distributed within the cell is considered. A lower bound for the multiple access channel capacity is provided under the assumption of constant receive power per user terminal, channel state information (CSI) at the receiver and with or without CSI at the transmitter.

A different research stream focuses on extended multicell networks with cooperating multiple antenna base stations. In this context, random matrix theory (RMT) resulted as a very powerful investigation tool. The analysis of these systems dates back to the works of Zaidel et al. [12] and it is based on a simple one-dimensional linear network referred in literature as Wyner network. The Wyner model is also adopted in most of the subsequent contributions based on RMT, e.g. in [13] where the benefit of cooperation on a multi-cell MIMO network is studied as the number of users and antennas at the base station grow large, in [14] where a more refined model is considered with directional antennas and finite number of users and antennas at the cooperating base stations. Critical weaknesses of the existing large system analysis based on RMT are due to the fact that the models do not account for the geographical random distribution of transmit and receive wireless nodes. A two-dimensional network model with randomly distributed nodes was proposed in [16] and targeted the optimization of the transmit power. The randomness in the system is obtained by locating transmit and receive pairs randomly in a lattice. Coherent potential approximation, a technique applied in statistical mechanics, is applied.

In this work we aim to combine the two key aspects of distributed antennas and extended cooperative networks and analyze the fundamental limits of this channel under the assumption of constant transmit power, knowledge of CSI at the centralized processing unit and no CSI at the transmitters. The analysis is carried out by extending the concept of ERM to two sets of independent random points (in our context the sets of transmitters and receivers) in an Euclidian space. Euclidian matrices are large group of random matrices introduced in [17] and found applications in various fields of physics. Given a random set in an Euclidian space $\mathcal{E}$ and a deterministic function $f : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$, the entries of an ERM realization consists of the values of $f(\cdot,\cdot)$ on all possible pairs of elements of the random set. A detailed overview can be found in [18] and [19] and several techniques to approximate the eigenvalue distributions and spectral properties of ERM are proposed. Up to the author's knowledge, this work presents the first extension of ERM theory to two independent sets of points as well as the first application of ERM theory to telecommunication systems. We apply an approximated decomposition of ERM which enables the application of free probability theory [20]. Based on this approximation, we determine an approximated expression of the average maximum achievable rate per unit area of an extended network in $\mathbb{R}^2$. To account for the randomness of the network topology, transmit and receive antennas are modeled as independent and homogeneously distributed over the entire area with given intensities.

II. System Model

We consider a wireless system consisting of infinite transmit and receive nodes in the Euclidean space $\mathbb{R}^2$. Both transmit and receive nodes are independently and homogeneously distributed over a squared lattice[1] of spacing $\tau > 0$. The receive antennas are connected to a processing unit (e.g. via

[1]Note that this is a technical assumption and the spacing τ can be chosen arbitrarily small. In the following, it will be clear that τ is requested to be sufficiently small to satisfy the sampling theorem for a specific function in $\mathbb{R}^2$.

optical fibers) such that decoding is performed jointly and optimally. The link between each transmitter-receiver pair is characterized by a pathloss attenuation dependent on the Euclidean distance between transmit and receive nodes. Let $\mathbf{r}_i = (r_{x,i}, r_{y,i})$ and $\mathbf{t}_j = (t_{x,j}, t_{y,j})$ be the Euclidean coordinates of receiver i and transmitter j on the square grid. Then, the pathloss is given by

$$f(\mathbf{r}_i, \mathbf{t}_j) = f(\|\mathbf{r}_i - \mathbf{t}_j\|_2) \quad (1)$$

where $\|\mathbf{r}_i - \mathbf{t}_j\|_2$ denotes the Euclidean distance between the two nodes. For the applicability of the mathematical tools proposed in the following section, $f(\cdot,\cdot)$ is required to vanish at the boundary of a finite disc (assumption satisfied in physical systems) and to satisfy the conditions of existence of a Fourier transform. In order to keep the presentation insightful and the computation simple, we follow the approach in [21] and model the pathloss as an exponentially decaying function

$$f(\mathbf{r}_i, \mathbf{t}_j) = \mathrm{e}^{-k_0\|\mathbf{r}_i - \mathbf{t}_j\|_2} \quad (2)$$

where $k_0 > 0$ is a positive constant. The transmit nodes do not have knowledge of the channel and transmit at the same power P. The receivers are impaired by additive white Gaussian noise with variance σ^2. The transmit signal to noise ratio (SNR) is denoted with $\rho = \frac{P}{\sigma^2}$. Then, the signal received at the discrete time interval t by receiver j is given by

$$y_i(t) = \sum_j P f(\|\mathbf{r}_i - \mathbf{t}_j\|_2) x_j(t) + w_i(t) \quad (3)$$

where $x_i(t)$ is the unitary energy symbol transmitted by node j; and $w_i(t)$ is the additive white Gaussian noise at receive node i.

Throughout this paper, vectors and matrices are denoted by bold lower case and bold capital letters; $\cdot^H$ denotes the Hermitian operator; the inner product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is shortly indicated by juxtaposition, i.e. $\mathbf{uv}$; $\delta(\mathbf{u})$ is the indicator function equal to 1 if $\mathbf{u} = \mathbf{0}$ and zero otherwise; $\|\mathbf{u}\|_2$ is the Euclidean norm of vector $\mathbf{u}$.

III. Mathematical tools and Performance Analysis

The performance analysis is obtained considering a finite system over a finite region and then determining its asymptotic performance assuming constant intensities of transmitters and receivers per unit area in the limit as the area and the number of nodes grows large. Thus, it is convenient to define more accurately the finite model.

Let $\mathcal{A}_L$ be a squared box of side L and area $A = L^2$ in $\mathbb{R}^2$. More specifically, we assume that $\mathcal{A}_L = \left[-\frac{L}{2}, +\frac{L}{2}\right) \times \left[-\frac{L}{2}, +\frac{L}{2}\right)$. Then, we choose a τ such that $L = \tau\theta$ with θ positive even[2] integer and we consider the points on a regular lattice $\mathbf{w} \equiv (\tau(\overline{w}_x + 1/2), \tau(\overline{w}_y + 1/2))$ with $\overline{w}_x, \overline{w}_y = -\frac{\theta}{2}, -\frac{\theta}{2}+1, \ldots \frac{\theta}{2}-1$. We denote by $\mathcal{A}_L^\sharp$ the lattice in $\mathcal{A}_L$. We model the distributed transmit and receive antennas as homogenous Bernoulli lattice processes $\Phi^{\mathcal{T}}_{\mathcal{A}_L^\sharp}$ and $\Phi^{\mathcal{R}}_{\mathcal{A}_L^\sharp}$ with parameters $\gamma_T = \rho_T\tau^2$ and $\gamma_R = \rho_R\tau^2$, respectively. It is worth noticing that two sequences of denser and denser Bernoulli lattice processes with constant intensity ρ_T and ρ_R converges in distribution to limiting Poisson point processes [7]. Additionally, let $\mathcal{T} = \{\mathbf{t}_j\}$ and $\mathcal{R} = \{\mathbf{r}_i\}$ be two realizations of the two Bernoulli lattice processes in $\mathcal{A}_L^\sharp$. They have cardinality N_T and N_R, respectively. Then $\rho_T = \frac{\mathrm{E}\{N_T\}}{\tau^2\theta^2}$ and $\rho_R = \frac{\mathrm{E}\{N_R\}}{\tau^2\theta^2}$, respectively. Corresponding to the sets $\mathcal{T}$ and $\mathcal{R}$ we define the random matrix $\mathbf{F}$ whose (i,j) element is the value of a deterministic function $f(\cdot,\cdot)$ in $\mathbb{R}^2 \times \mathbb{R}^2$, i.e.

$$\mathbf{F} = f(\mathbf{r}_i, \mathbf{t}_j). \quad (4)$$

To study the statistical properties of matrix $\mathbf{F}$ it is convenient to express function $f(\cdot,\cdot)$ as the expansion of orthogonal functions ψ_α in $\mathcal{A}_L^\sharp$. This approach has the advantage of decoupling the effects of the randomness of the matrix from the complexity of the function $f(\cdot,\cdot)$. Throughout this work, we adopt as orthogonal set of functions on $\mathcal{A}_L^\sharp$

$$\psi_{\boldsymbol{\ell}}^{(L)}(\mathbf{w}) = \frac{1}{\theta}\exp\left(+i\omega_L \boldsymbol{\ell}\mathbf{w}\right) \quad (5)$$

where $\omega_L = \frac{2\pi}{L}$, $\boldsymbol{\ell} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ is a two dimensional vector of integers and $\mathbf{w} \in \mathcal{A}_L^\sharp$. It is straightforward to verify that these functions are orthogonal on $\mathcal{A}_L^\sharp$, i.e.

$$\sum_{\mathbf{w}\in\mathcal{A}_L^\sharp} \psi_{\boldsymbol{\ell}}^{(L)}(\mathbf{w})\psi_{\mathbf{m}}^{(L)*}(\mathbf{w}) = \begin{cases} 0, & \boldsymbol{\ell} \neq \mathbf{m} \\ 1, & \boldsymbol{\ell} = \mathbf{m}. \end{cases} \quad (6)$$

Additionally, we assume that the orthogonal set satisfies the property

$$\sum_{\mathbf{w}\in\mathcal{A}_L^\sharp} \psi_{\boldsymbol{\ell}}^{(L)}(\mathbf{w}) = 0. \quad (7)$$

This property is satisfied for all pairs of integers (ℓ_x, ℓ_y) with $\ell_x, \ell_y \neq \pm n\theta$ and $n \in \mathbb{N}_0$ for the function set defined in (5). Then, we define the discrete transform

$$\begin{aligned} T_{\boldsymbol{\ell},\mathbf{m}}^{(L)} &= \sum_{\mathbf{r}\in\mathcal{A}_L^\sharp}\sum_{\mathbf{t}\in\mathcal{A}_L^\sharp} f(\mathbf{r},\mathbf{t})\psi_{\boldsymbol{\ell}}^{(L)*}(\mathbf{r})\psi_{\mathbf{m}}^{(L)}(\mathbf{t}) \\ &= \frac{1}{\theta^2}\sum_{\mathbf{r}\in\mathcal{A}_L^\sharp}\sum_{\mathbf{t}\in\mathcal{A}_L^\sharp} f(\mathbf{r},\mathbf{t})\mathrm{e}^{-i\omega_L(\boldsymbol{\ell}\mathbf{r} - \mathbf{m}\mathbf{t})}. \end{aligned} \quad (8)$$

It is straightforward to verify that

$$f(\mathbf{r},\mathbf{t}) = \sum_{\substack{\ell_x,\ell_y,m_x,m_y=\{0,1,\ldots\theta-1\} \\ (\ell_x,\ell_y)\neq(0,0) \\ (m_x,m_y)\neq(0,0)}} T_{\boldsymbol{\ell},\mathbf{m}}^{(L)}\psi_{\boldsymbol{\ell}}^{(L)}(\mathbf{r})\psi_{\mathbf{m}}^{(L)*}(\mathbf{t}).$$

Then, the matrix $\mathbf{F}^{(L)}$ can be written as

$$\mathbf{F}^{(L)} = \mathbf{\Psi}_R^{(L)}\mathbf{T}^{(L)}\mathbf{\Psi}_T^{(L)H} \quad (9)$$

where $\mathbf{T}^{(L)}$ is a $(\theta^2-1)\times(\theta^2-1)$ matrix with elements $T_{\boldsymbol{\ell},\mathbf{m}}^{(L)}$, $\mathbf{\Psi}_R^{(L)}$ is an $N_R \times (\theta^2-1)$ matrix with element $(j,\boldsymbol{\ell})$ $\psi_{j,\boldsymbol{\ell}}^{(L)} = \psi_{\boldsymbol{\ell}}^{(L)}(\mathbf{r}_j)$ and $\mathbf{\Psi}_T^{(L)}$ is an $N_T \times (\theta^2-1)$ matrix with element $(k,\mathbf{m})$ $\psi_{k,\mathbf{m}}^{(L)} = \psi_{\mathbf{m}}^{(L)}(\mathbf{t}_k)$. When the area of $\mathcal{A}^{(L)}$ increases θ, N_R and N_T also increase according to the following relations

$$N_R = \rho_R\tau^2\theta^2, \qquad N_T = \rho_T\tau^2\theta^2, \qquad N_R = \frac{\rho_R}{\rho_T}N_T. \quad (10)$$

In order to determine the asymptotic eigenvalue distribution of the matrix $\mathbf{F}^{(L)H}\mathbf{F}^{(L)}$ as $L \to +\infty$ it is essential to

[2] The assumption to be even is made to simplify the notation but it is not strictly required.

characterize the matrix $\mathbf{T}^{(L)}$ as $L \to +\infty$ and determine its asymptotic eigenvalue distribution. The following proposition summarizes the asymptotic properties of $\mathbf{T}^{(L)}$ when the matrix $\mathbf{F}^{(L)}$ is defined over the exponentially decaying function in (2).

PROPOSITION 1 *Let $\epsilon > 0$ be an arbitrary small positive value and let $\tau \leq \frac{\pi^2\epsilon}{2k_0}$. Then, the asymptotic eigenvalue density function of matrix $\mathbf{T}^{(L)}$ as $L \to +\infty$ is given by*

$$f_{\mathbf{T}}(x) = \frac{(2\pi k_0 \tau)^{2/3}}{6\pi} x^{-5/3} \qquad \frac{2\pi k_0 \tau}{(\sqrt{k_0^2\tau^2+\pi^2})^3} \leq x \leq \frac{2\pi}{k_0^2\tau^2}$$

and can be effectively approximated elsewhere by $f_{\mathbf{T}} = \left(1-\frac{\pi}{4}\right)\delta(x-\eta)$ where η is a positive constant in the interval[3] $\left[\frac{2\pi k_0\tau}{(\sqrt{k_0^2\tau^2+2\pi^2})^3}, \frac{2\pi k_0\tau}{(\sqrt{k_0^2\tau^2+\pi^2})^3}\right]$.

The results of this proposition are derived in Appendix A.

In order to obtain an approximate expression of the eigenvalue distribution $f_{\mathbf{F}^H\mathbf{F}}(\lambda)$ of matrix $\mathbf{F}^{(L)H}\mathbf{F}^{(L)}$ as $L \to +\infty$ or, equivalently, $N_R, N_T, \theta^2 \to +\infty$ with constant ratios (10) we approximate the matrices $\mathbf{\Psi}_R^{(L)}$ and $\mathbf{\Psi}_T^{(L)}$ by Gaussian matrices of i.i.d. zero mean elements with variance θ^{-2}. Note that this approximation has been widely applied both in telecommunication systems (e.g. [22]) and mesoscopic physics (e.g. [19], [23]).

Under this approximation, we can apply the following lemma.

LEMMA 1 *Let $\widetilde{\mathbf{\Phi}}_T$ and $\widetilde{\mathbf{\Phi}}_R$ be Gaussian matrices of i.i.d. zero mean elements with variance θ^{-2} and size $\theta^2 \times N_T$ and $\theta^2 \times N_R$, respectively. Let $\mathbf{A}$ be a $\theta^2 \times \theta^2$ diagonal matrix with eigenvalue probability density function that converges to $f_{\mathbf{A}}(x)$, with $x \in \mathbb{R}$, as $\theta^2 \to +\infty$. Then, as θ^2, N_T, $N_R \to +\infty$ with constant ratios $\frac{N_R}{\theta^2} \to \gamma_R$ and $\frac{N_T}{\theta^2} \to \gamma_T$, the asymptotic eigenvalue distribution $f_{\mathbf{C}}(x)$ of the matrix*

$$\mathbf{C} = \widetilde{\mathbf{\Phi}}_T \mathbf{A} \widetilde{\mathbf{\Phi}}_R^H \widetilde{\mathbf{\Phi}}_R \mathbf{A} \widetilde{\mathbf{\Phi}}_T^H$$

obeys

$$sG_{\mathbf{C}}(s) + 1 = -G_{\mathbf{C}}(s)\left(\gamma_T s G_{\mathbf{C}}(s) + \gamma_T - \gamma_R\right) \int \frac{x^2 f_{\mathbf{A}}(x)\mathrm{d}x}{1 - \gamma_T G_{\mathbf{C}}(s)\left(\gamma_T s G_{\mathbf{C}}(s) + \gamma_T - \gamma_R\right)x^2} \quad (11)$$

being $G_{\mathbf{C}}(s)$ the Stieltjes transform of $f_{\mathbf{C}}(x)$, i.e.

$$G_{\mathbf{C}}(s) = \int \frac{f_{\mathbf{C}}(x)\mathrm{d}x}{x-s}. \quad (12)$$

The derivation of this result is omitted since it follows along the lines of the derivation in [24], Appendix A. An implicit expression of $G_{\mathbf{F}^H\mathbf{F}}(s)$, the Stieltjes transform of $f_{\mathbf{F}^H\mathbf{F}}(\lambda)$ is obtained by computing (11) for $f_{\mathbf{A}}(x) = f_{\mathbf{T}}(x)$, the eigenvalue probability density function provided in Proposition 1.

Under the constraint of constant transmit power, the average maximum achievable rate per unit area and frequency band for a network over a finite lattice $\mathcal{A}_L^{\#}$ are related to the spectrum of the matrix $\mathbf{F}$ by

$$C(\rho, \mathbf{F}^L, N_R, N_T) = \frac{1}{2\theta^2\tau^2}\sum_{n=1}^{N_T}\log_2\left(1+\rho\lambda_n(\mathbf{F}^{(L)H}\mathbf{F})^{(L)}\right)$$

where $\lambda_n(\cdot)$ denotes the n-th eigenvalue of the matrix argument. Asymptotically, as $L \to +\infty$ we can resort the well known relation between achievable rate per unit area and $G_{\mathbf{F}^H\mathbf{F}}(s)$, the Stieltjes transform of the asymptotic eigenvalues density function of the channel covariance matrix $\mathbf{F}^H\mathbf{F}$, (see e.g. [10], [25]–[27])

$$\mathcal{C}(\rho, \rho_R, \rho_T) = \frac{\rho_T}{2\ln 2}\int_0^\rho v^{-1}\left(1 - v^{-1}G_{\mathbf{F}^H\mathbf{F}}(-v^{-1})\right)\mathrm{d}v$$

and assume $G_{\mathbf{F}^H\mathbf{F}}(s) \approx G_{\mathbf{C}}(s)$ to obtain and approximate expression of asymptotic average achievable rate per unit area.

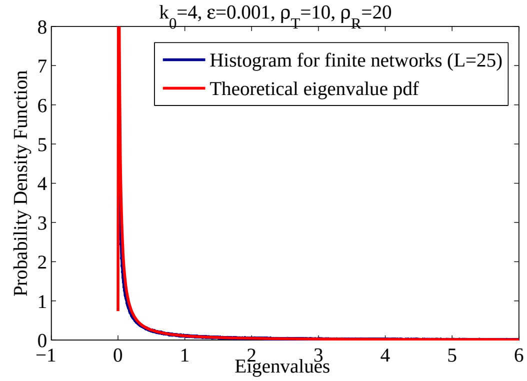

Fig. 1. Asymptotic eigenvalue distribution of the matrix $\mathbf{F}^H\mathbf{F}$ in solid red line for $\rho_T = 20$, $\rho_R = 40$, and $k_0 = 4$. The asymptotic eigenvalue distribution is compared to the empirical eigenvalue distribution given by the histogram in blue.

IV. Simulation Results

In this section we compare the analytical results with the empirical results obtained by simulating a network of finite size. We consider systems with the pathloss by $k_0 = 4$. In Figure 1 we assume $\rho_T = 20$ and $\rho_R = 40$ and compare the theoretical eigenvalue pdf to the empirical distribution. Although the two curves match quite well in the central part they present some mismatch in the extremes of the support: the empirical distribution shows an higher concentration of eigenvalues close to zero while the theoretical eigenvalue distribution presents a larger support. In Figure 2, we consider a system with transmitter intensity $\rho_T = 20$ while the intensity of the receivers varies in the range $\rho_R = [20, 200]$. The curves show the capacity per unit frequency band per unit area when the number of the receivers' intensity increases for $E_b/N_0 = 3$ dB and $E_b/N_0 = 8$ dB. At low values of E_b/N_0 the analytical approximation is very tight while it becomes looser when E_b/N_0 increases. Imbalances between the number of transmit and receive antenna densities is also an additional factor that influences the tightness of the analytical bound.

Acknowledgements

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[3]Because of the constraints on τ for ϵ arbitrarily small also the values on this interval are close to zero.

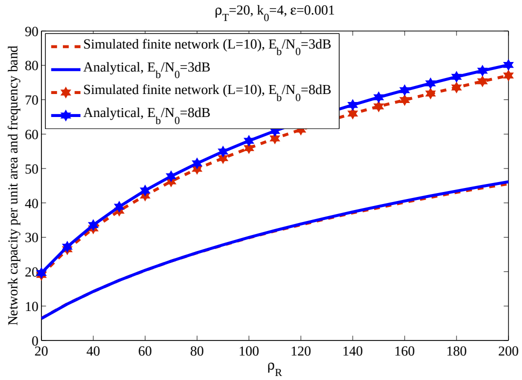

Fig. 2. Capacity per unit frequency band per unit area versus density of receive antennas. Density of transmit antennas constant $\rho_T = 20$, exponential decaying constant $k_0 = 4$. Solid and dashed lines show the analytical bound and the actual capacity in a finite network of area 100.

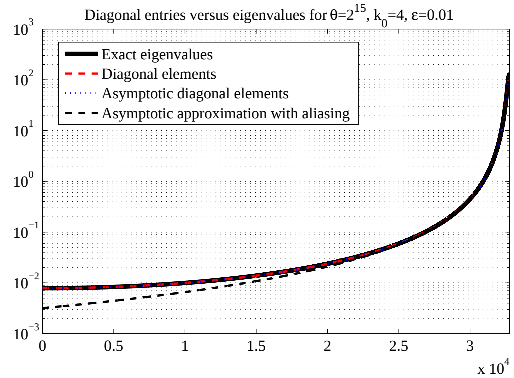

Fig. 3. The black solid line shows the eigenvalues of matrix $\mathbf{T}^{(L)}$ for $\theta = 2^{15}$ and $k_0 = 4$. The read dashed line and the blue dotted line show the diagonal elements of the finite and the asymptotic matrix. Finally, the dashed black line shows the approximation affected by aliasing of the asymptotic diagonal elements.

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Appendix

Before considering the 2-dimensional (2D) system model we consider the 1-dimensional (1D) one. In this case, the lattice is defined over the segment $\widetilde{\mathcal{A}} = [-L/2, L/2]$ and consists of the points $w \equiv \tau(\overline{w}_x + 1/2)$ with $\overline{w}_x$ defined as in the 2D case. The two Bernoulli lattice points $\Phi^{\widetilde{\mathcal{T}}}_{\widetilde{\mathcal{A}}^\sharp}$ and $\Phi^{\widetilde{\mathcal{R}}}_{\widetilde{\mathcal{A}}^\sharp}$ are characterized by the parameters $\widetilde{\gamma}_T = \widetilde{\rho}_T \tau$ and $\widetilde{\gamma}_R = \widetilde{\rho}_R \tau$ and, if $\widetilde{N}_T$ and $\widetilde{N}_R$ is the cardinality of the realizations of the two processes, then $\widetilde{\rho}_T = \frac{\mathrm{E}\{\widetilde{N}_T\}}{\tau\theta}$ and $\widetilde{\rho}_R = \frac{\mathrm{E}\{\widetilde{N}_R\}}{\tau\theta}$. Definitions similar to the 2D case hold for the system model (3) and the matrix decomposition (9) with the difference that the orthogonal set of functions in $\widetilde{\mathcal{A}}^\sharp$ is defined as $\widetilde{\psi}_\ell = \frac{1}{\sqrt{\theta}} \mathrm{e}^{+\imath\omega_L \ell w}$. The discrete transform $\widetilde{T}^L_{\ell,m}$ in the 1D case admits a closed form expression even for finite L thanks to the existence of well known closed form expressions for $\sum_{r=-\theta/2}^{\theta/2-1} \mathrm{e}^{(-\alpha+\imath\beta)r}$. Due to space constraints, we omit the derivation and provide directly the diagonal elements of $\widetilde{\mathbf{F}}^{(L)}$ in (13) at the top of next page, with $\nu = \frac{2\pi\ell}{\theta}$. Similarly, we can obtain the out-diagonal elements $\widetilde{T}^{(L)}_{\ell,m}$ provided in (14) on the top of next page. It is apparent that $\widetilde{T}^{(L)}_{\ell,m} \to 0$ as $\theta \to +\infty$ with rate θ^{-1}. However, there are ℓ such that $\lim_{\theta\to\infty} \sum_{m:m\neq\ell} \left|\widetilde{T}^{(L)}_{\ell,m}\right|$ does not converge to zero. Then, we cannot adopt the classical argument based on the Gershgorin circle theorem to prove the convergence of the eigenvalues of the matrix $\mathbf{T}^{(L)}$ to its diagonal elements. Nevertheless, $\lim_{\theta\to\infty} \sum_{m:m\neq\ell} \widetilde{T}^{(L)}_{\ell,m} \to 0$ and numerous simulations show a perfect match between the eigenvalues of the matrix $\mathbf{F}^{(L)}$, as shown in Fig. 3. We leave the analysis of this aspect for further studies and assume that the same property holds for the 2D case. In the following we focus on the derivation of the diagonal elements of the matrix $\mathbf{T}^{\infty}$ for the 2D case. For finite $L = \tau\theta$ and $\mathbf{r} = \tau\mathbf{R}$, $\mathbf{t} = \tau\mathbf{T}$ with $\mathbf{r}, \mathbf{t} \in \mathcal{A}^\sharp_L$ and $\boldsymbol{\nu} = \boldsymbol{\mu}$, where $\boldsymbol{\nu} = \omega_L \boldsymbol{\ell}$ and $\boldsymbol{\mu} = \omega_L \boldsymbol{m}$,

$$T^{(L)}_{\boldsymbol{\nu},\boldsymbol{\nu}} = \frac{1}{\theta^2} \sum_{\mathbf{T}} \sum_{\mathbf{R}} \mathrm{e}^{-k_0\tau|\mathbf{R}-\mathbf{T}|} \mathrm{e}^{-\imath\boldsymbol{\nu}(\mathbf{R}-\mathbf{T})}$$

Observe that the diagonal elements $T^{(L)}_{\boldsymbol{\nu},\boldsymbol{\nu}}$ depend only on $\boldsymbol{\Delta} = \mathbf{R} - \mathbf{T}$ and our problem reduces to determine

$$T(\boldsymbol{\nu}) = \lim_{\theta\to+\infty} \sum_{\tau\boldsymbol{\Delta}\in\mathcal{A}^\sharp_L} \mathrm{e}^{-k_0\tau|\boldsymbol{\Delta}|} \mathrm{e}^{-\imath\boldsymbol{\nu}\boldsymbol{\Delta}} \tag{15}$$

for $\boldsymbol{\nu} \in [-\pi, +\pi) \times [-\pi, +\pi)$. In order to compute the function $T(\boldsymbol{\nu})$, let us observe that $T(\boldsymbol{\nu})$ is the 2-dimensional (2D) Fourier representation of a discrete 2D sequence obtained by sampling the function $\mathrm{e}^{-k_0\sqrt{x^2+y^2}}$ in the point of the set $\mathcal{A}^\sharp_L$. As well known (e.g. [28]), this frequency response $T(\boldsymbol{\nu})$ is related to the Fourier transform $\mathcal{T}(\boldsymbol{\Omega})$ of the continuous 2-dimensional function $f(\mathbf{r}, \mathbf{t})$

$$\mathcal{T}(\boldsymbol{\Omega}) = \int_{\mathbb{R}^2} \mathrm{e}^{-k_0\sqrt{x^2+y^2}} \mathrm{e}^{-\imath(\Omega_x x + \Omega_y y)} \mathrm{d}x\mathrm{d}y$$

by the relation

$$T(\boldsymbol{\nu}) = \frac{1}{\tau^2} \sum_{r=-\infty}^{+\infty} \sum_{s=-\infty}^{+\infty} \mathcal{T}\left(\frac{\nu_x}{\tau} + \frac{2\pi r}{\tau}, \frac{\nu_y}{\tau} + \frac{2\pi s}{\tau}\right). \tag{16}$$

Thus, the problem boils down to determine $\mathcal{T}(\boldsymbol{\Omega})$. From (16)

$$\begin{aligned}\mathcal{T}(\boldsymbol{\Omega}) &= \int_{\mathbb{R}^2} \mathrm{e}^{-k_0\sqrt{x^2+y^2}} \mathrm{e}^{-\imath(\Omega_x x + \Omega_y y)} \mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty \rho \mathrm{e}^{-k_0\rho} \mathrm{d}\rho \int_0^{2\pi} \mathrm{e}^{-\imath\rho\rho_\Omega \cos(\phi-\phi_\Omega)} \mathrm{d}\phi\end{aligned}$$

where $\rho = \sqrt{x^2+y^2}$, $\phi = \arctan\frac{y}{x}$, $\rho_\Omega = \sqrt{\Omega_x^2 + \Omega_y^2}$, and $\phi_\Omega = \arctan\frac{\Omega_y}{\Omega_x}$. By making use of the Bessel function identity

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\imath x\cos(\phi-\theta)} \mathrm{d}\theta \tag{17}$$

where $J_0(x)$ is the Bessel function of first kind, we obtain

$$\mathcal{T}(\boldsymbol{\Omega}) = 2\pi \int_0^{+\infty} \rho \mathrm{e}^{-k_0\rho} J_0(\rho_\Omega \rho) \mathrm{d}\rho = \frac{2\pi k_0}{(k_0^2 + \Omega_x^2 + \Omega_y^2)^{3/2}}$$

$$T_{\ell,\ell}^{(L)} = 1 - \frac{2(\mathrm{e}^{-k_0\tau(\theta-1)}(\mathrm{e}^{k_0\tau} - \cos(\nu)) - \mathrm{e}^{k_0\tau}\cos(\nu) + 1)}{\mathrm{e}^{2k_0\tau} - 2\mathrm{e}^{k_0\tau}\cos(\nu) + 1} + \frac{2\mathrm{e}^{k_0\tau}(2\mathrm{e}^{k_0\tau} - \cos(\nu) - \mathrm{e}^{2k_0\tau}\cos(\nu))}{\theta(\mathrm{e}^{2k_0\tau} - 2\mathrm{e}^{k_0\tau}\cos(\nu) + 1)^2} + \frac{2\mathrm{e}^{-k_0\tau(\theta-1)}(\mathrm{e}^{k_0\tau} - \cos(\nu))}{(\mathrm{e}^{2k_0\tau} - 2\mathrm{e}^{k_0\tau}\cos(\nu) + 1)} + \frac{2\mathrm{e}^{-k_0\tau(\theta-1)}(\mathrm{e}^{2k_0\tau}\cos(\nu) - 2\mathrm{e}^{k_0\tau} + \cos(\nu))}{\theta(\mathrm{e}^{2k_0\tau} - 2\mathrm{e}^{k_0\tau}\cos(\nu) + 1)^2} \quad (13)$$

$$T_{\ell,m}^{(L)} = \frac{\cos(\pi m)\cos(\pi\ell)\cos(\pi(\ell-m)/\theta)(1-\mathrm{e}^{-k_0\tau\theta})(4\cos(\pi(\ell-m)/\theta) - 2(\mathrm{e}^{k_0\tau} + \mathrm{e}^{-k_0\tau})\cos(\pi(\ell+m)/\theta))}{\theta\left((2\cos(\pi(\ell-m)/\theta) - (\mathrm{e}^{k_0\tau} + \mathrm{e}^{-k_0\tau})\cos(\pi(\ell+m)/\theta))^2 + (\mathrm{e}^{-k_0\tau} - \mathrm{e}^{k_0\tau})^2\sin(\pi(\ell+m)/\theta)^2\right)} \quad (14)$$

and by applying (16)

$$T(\boldsymbol{\nu}) = 2\pi k_0\tau \sum_r \sum_s \left(k_0^2\tau^2 + (\nu_x + 2\pi r)^2 + (\nu_y + 2\pi s)^2\right)^{-3/2}$$

Note that for τ sufficiently small, i.e. such that, for an arbitrarily small $\epsilon > 0$,

$$\mathcal{T}(\pm\pi, \pm\pi) = \frac{2\pi k_0\tau}{(\tau^2k_0^2 + \pi^2)^{3/2}} \leq \epsilon \approx 0$$

or more conservatively for

$$\tau \leq \frac{\pi^2\epsilon}{2k_0}$$

we can neglect aliasing effects and the diagonal elements of the matrix $\mathbf{T}$ in the limit for $\theta \to +\infty$, can be approximated by

$$T(\boldsymbol{\nu}) = \frac{2\pi k_0\tau}{\left(\sqrt{\tau^2k_0^2 + \mathbf{u}_x^2 + \nu_y^2}\right)^3} \qquad |\nu_x| < \pi \text{ and } |\nu_y| < \pi.$$

Fig. 3 shows the mismatch between the actual eigenvalues and the asymptotic diagonal elements obtained by neglecting the aliasing.

In order to utilize the spectrum of the matrix $\mathbf{T}$ in the framework of random matrix theory we need to express it in terms of probability distribution function $F_{\mathbf{T}}(x)$ which denotes fraction of eigenvalues of matrix $\mathbf{T}$ nongreater than x. To determine $F_{\mathbf{T}}(x)$ let us observe that

$$F_{\mathbf{T}}(x) = \begin{cases} 0, \text{ for } 0 \leq x < \eta; \\ 1 - \frac{\pi}{4}, \text{ for } \eta \leq x < \frac{2\pi k_0\tau}{(\sqrt{k_0^2\tau^2+\pi^2})^3}; \\ 1 - \frac{1}{4\pi}\left(\frac{2\pi k_0\tau}{x}\right)^{2/3} + \frac{k_0^2\tau^2}{4\pi} \\ \quad \text{for } \frac{2\pi k_0\tau}{(\sqrt{k_0^2\tau^2+\pi^2})^3} \leq x \leq \frac{2\pi}{k_0^2\tau^2} \end{cases} \quad (18)$$

This yields the asymptotic eigenvalue density function of matrix $\mathbf{T}$ in Proposition 1.

References

[1] A. A. M. Saleh, A. Rustako, and R. Roman, "Distributed antennas for indoor radio communications," *IEEE Transactions on Communications*, vol. 35, no. 12, pp. 1245–1251, Dec. 1987.

[2] K. Kerpez, "A radio access system with distributed antennas," *IEEE Transactions on Vehicular Technology*, vol. 45, no. 2, pp. 265–275, May 1996.

[3] W. Choi and J. Andrews, "Downlink performance and capacity of distributed antenna systems in a multicell environment," *IEEE Transactions on Wireless Communications*, vol. 6, no. 1, pp. 69–73, Jan. 2007.

[4] R. Heath, T. Wu, Y. Kwon, and A. Soong, "Multiuser MIMO in distributed antenna systems with out-of-cell interference," *IEEE Transactions on Signal Processing*, vol. 59, no. 10, pp. 4885–4899, Oct. 2011.

[5] J. Zhang and J. Andrews, "Distributed andtenna systems with randomness," *IEEE Transactions on Wireless Communications*, vol. 7, no. 9, pp. 3636–3646, Sep. 2008.

[6] Y. Lin and W. Yu, "Ergodic capacity analysis of downlink distributed antenna systems using stochastic geometry," in *Proc. of IEEE International Conference on Communications (ICC)*, 2013, pp. 1931–1936.

[7] M. Haenggi, *Stochastic Geometry for Wireless Networks*. Cambridge University Press, 2012.

[8] J. Gan, Y. Li, L. Xiao, S. Zhou, and J. Wang, "On sum rate and power comsumption of multiuser distributed antenna system with circular antenna layout," *EURASIP Journal on Wireless Communications and Networking*, vol. 2007, Jul. 2007.

[9] W. Feng, Y. Li, S. Zhou, J. Wang, and M. Xia, "Uplink sum capacity evaluation of multi-user distributed antenna systems," *Elsevier, International Journal of Electronics and Communications*, vol. 64, pp. 168–172, Feb. 2010.

[10] R. Couillet and M. Debbah, *Random Matrix Methods for Wireless Communications*. Cambridge University Press, 2011.

[11] L. Dai, "A comparative study on uplink sum capacity with co-located and distributed antennas," *IEEE Journal on Selected Areas in Communications*, vol. 29, no. 6, pp. 1200–1213, Jun. 2011.

[12] B. M. Zaidel, S. Shamai, and S. Verdú, "Multicell uplink spectral efficiency of coded DS-CDMA with random signatures," *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 8, pp. 1556–1569, Aug. 2001.

[13] D. Aktas, M. Bacha, J. Evans, and S. Hanly, "Scaling results on the sum capacity of cellular networks with mimo links," *IEEE Transactions on Information Theory*, vol. 52, no. 7, pp. 3264 – 3274, july 2006.

[14] N. Levy, O. Zeitouni, and S. Shamai, "On information rates of the fading wyner cellular model via the thouless formula for the strip," *Information Theory, IEEE Transactions on*, vol. 56, no. 11, pp. 5495 –5514, nov. 2010.

[15] O. Somekh, O. Simeone, Y. Bar-Ness, A. M. Haimovich, U. Spagnolini, and S. Shamai, "An information theoretic view of distributed antenna processing in cellular systems," in *Distributed Antenna Systems: Open Architecture for Future Wireless Communications*, 2007, pp. 31–64.

[16] A. L. Moustakas and N. Bambos, "Power optimization on a random wireless network," in *Proc. of IEEE International Symposium on Information Theory (ISIT)*, Istambul, Turkey, Aug. 2013.

[17] M. Mézard, G. Parisi, and A. Zee, "Spectra of euclidean random matrices," *Journal of Physics B: Nuclear Physics*, 1999.

[18] A. Goetschy and S. Skipetrov, "Non-hermitian euclidean random matrix theory," *Phys. Rev. E*, Jun. 2011, arxiv:1102.1850v2.

[19] ——, "Euclidean random matrices and their applications in physics," Mar. 2013, arxiv:1303.2880v1.

[20] F. Hiai and D. Petz, *The Semicircle Law, Free Random Variables and Entropy*. Providence, RI: American Mathematical Society, 2000.

[21] Abdallah and M. Debbah, "Downlink CDMA: to cell or not to cell," in *Proceedings of the 12th European Signal Processing Conference (EUSIPCO'04)*, Vienna, Austria, May 2004.

[22] R. R. Müller, "On the asymptotic eigenvalue distribution of concatenated vector–valued fading channels," *IEEE Transactions on Information Theory*, vol. 48, no. 7, pp. 2086–2091, Jul. 2002.

[23] S. E. Skipetrov and A. Goetschy, "Eigenvalue distributions of large euclidean random matrices for waves in random media," Dec. 2010, arXiv preprint arXiv:1007.1379.

[24] R. Müller, L. Cottatellucci, and M. Vehkaperä, "Blind pilot decontamination," *Submitted to IEEE Journal on Selected Areas in Signal Processing*, Sep. 2013, on line: arXiv:1309.6806.

[25] S. Shamai (Shitz) and S. Verdú, "The impact of frequency–flat fading on the spectral efficiency of CDMA," *IEEE Transactions on Information Theory*, vol. 47, no. 4, pp. 1302–1327, May 2001.

[26] L. Cottatellucci, R. R. Müller, and M. Debbah, "Asynchronous CDMA systems with random spreading–part i: Fundamental limits," *IEEE Transactions on Information Theory*, vol. 56, no. 4, pp. 1477 – 1497, Apr. 2010.

[27] L. Cottatellucci and M. Debbah, "On the capacity of MIMO Rice channels," in *Proc. 42nd Allerton Conf. on Communication, Control and Computing*, Monticello, Illinois, Sep./Oct. 2004.

[28] A. V. Oppenheim and R. W. Schafer, *Digital Signal Processing*. New Jersey: Prentice–Hall, 1975.