On the Decision-Directed Equalization of Constant Modulus Signals

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Abstract

Decision-Directed (DD) equalization is the most primitive Blind Equalization (BE) method for the cancelling of Inter-Symbol-Interference (ISI) in data communication systems. Even though DD equalizers are believed to be unable to open the channel eye when it is initially closed, this does not seem to be true in the case of Constant-Modulus (CM) constellations (pure phase modulation). We investigate the shape of the DD cost function in this case and obtain several interesting results that indicate that the DD algorithm should be capable of opening a closed channel eye in the CM case. Based on this fact, we propose a novel hybrid CMA-DD equalization scheme that offers an appealing alternative to the Generalized Sato (GSATO) algorithm for QAM constellations. Our theoretical claims about the performance of DD equalizers as well as the performance of our novel scheme are verified through computer simulations.

1 Introduction

Consider the baseband representation of a QAM digital communication system, as depicted in figure 1. The channel is assumed to be a linear filter whose z-transform of its impulse response is $C(z^{-1})$ and the equalizer is also a FIR filter with corresponding z-transform $W(z^{-1})$. We denote by $\{a_i\}, \{x_i\}$ and $\{y_i\}$ the transmitted (input), received and equalized sequence of data, respectively. If the channel equalizer is assumed to have N taps, the equalizer output can be written as $y_k = X_k^H W_k$, where $X_k^H = [x_k \; x_{k-1} \cdots x_{k-N+1}]$ and W_k is a $N \times 1$ vector containing the equalizer taps at time instant k. We also denote by s the overall channel-equalizer impulse response : $s_i = c_i * w_i$ ($*$ denotes convolution). In terms of s, the equalizer output can be written as

$$y_k = s * a_k = \sum_{i=-\infty}^{\infty} s_i a_{k-i} \quad . \qquad (1)$$

In BE, the input sequence $\{a_i\}$ is identified based only on the received sequence $\{x_i\}$ and some a priori knowledge of the statistics of $\{a_i\}$. A strong identifiability

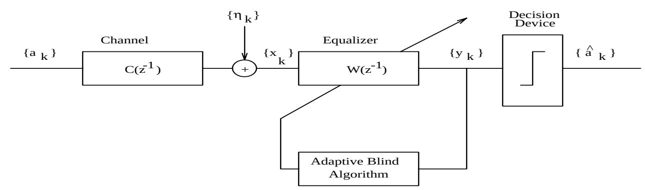

Figure 1: A typical blind equalization scheme

result has been given in [5], where it was stated that a necessary and sufficient condition for zero-forcing equalization in the noiseless case is that the following two conditions hold

$$\begin{cases} E\left(|y|^2\right) = E\left(|a|^2\right) \\ |K(y)| = |K(a)| \end{cases} , \qquad (2)$$

where $E(\cdot)$ denotes statistical expectation and $K(\cdot)$ is the *Kurtosis* of a process ($K(z_i) = E\left(|z_i|^4\right) - 2E^2\left(|z_i|^2\right) - \left|E(z_i^2)\right|^2$). It turns out in [5] that in the case of a sub-Gaussian ($K(a) < 0$) and symmetrical ($E(a^2) = 0$) input distribution, a noiseless channel and an infinite-length equalizer, the Godard cost function [1] defined as [1]

$$J^{(p)}(W) = \frac{1}{2p} E(|y_k|^p - r_p)^2 \quad p = 1, 2, \ldots, \qquad (3)$$

in the particular case $p = 2$ is a convex function of s whose global minima are optimal settings ($s = e^{j\theta}(\cdots 0 \; 1 \; 0 \cdots)$). This result explains why the popular CMA 2-2 algorithm [2], which is the member of the Godard algorithm

$$W_{k+1} = W_k + \mu X_k y_k |y_k|^{p-2}(|y_k|^P - r_p) \quad , \qquad (4)$$

corresponding to $p = 2$, has an optimal performance in the case of a noiseless channel and an infinite-length equalizer, even if $\{a_i\}$ is not drawn from a CM constellation, provided that it is sub-Gaussian and symmetrical.

[1] r_p is the dispersion constant defined as $r_p = E\frac{|a_k|^{2p}}{|a_k|^p}$

No similar result however seems to exist (up to our knowledge) for the CMA 1-2 algorithm (4) for $p = 1$. We are especially interested in the latter algorithm for the following reason: the CMA 1-2 can be written in the following way:

$$W_{k+1} = W_k + \mu X_k(y_k - r_1 sign(y_k)) \ , \qquad (5)$$

where $sign(y) = \frac{y}{|y|}$ denotes the projection of a complex scalar on the unit circle that maintains its angle. Comparing (5) to the DD algorithm given by

$$W_{k+1} = W_k + \mu X_k(y_k - dec(y_k)) \ , \qquad (6)$$

where $dec(y)$ denotes the closest constellation symbol to y, we notice that when the input signal is CM, the *error signals* at each iteration

$$\begin{cases} e_k^D &= y_k - dec(y_k) \\ e_k^C &= y_k - r_1 sign(y_k) \ , \end{cases} \qquad (7)$$

are close to one another, and therefore the two algorithms are similar in this case. Figure 2 shows the corresponding error signals for a 4-QAM constellation (fat-line segments). The more symbols on the circle of radius r_1, the closer the two errors will be, and in the asymptotic case of an infinity of constellation symbols on the circle the two algorithms are identical! This remark implies that if the CMA 1-2 algorithm has a good performance for CM constellations, the DD also performs similarly and should therefore be able to open an initially closed channel eye (contrary to what seems to be believed widely to date).

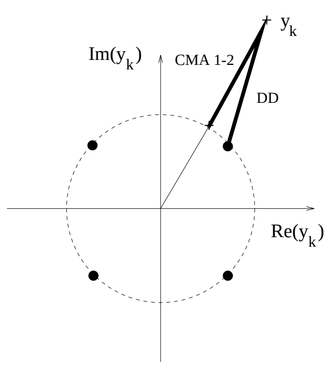

Figure 2: The similarity between CMA 1-2 and DD

The counterpart of the CMA 1-2 algorithm for PAM is the Sato algorithm [3], whose update equation is identical to (5), the only difference being that all the quantities are real. A strong result about the behaviour of the Sato algorithm can be found in [4], where it was stated that in the case of a continuous sub-Gaussian input distribution, a noiseless channel and an infinite equalizer length, the Sato cost function admits as only local (and global) minima the optimal settings for s. This result is less strong than the one given for the CMA 2-2 in [5], since it is only valid for continuous (and not for discrete) input distributions. However it gives valuable insight to the performance of the Sato algorithm, and one would expect that under similar conditions the CMA 1-2 should also have no local minima.

Based on these remarks, we will analyze the shape of the DD cost function in the case of a CM input, through a corresponding analysis of the CMA 1-2.

2 Analysis of the shape of the CMA 1-2 cost function

In the sequel we make the following assumptions:

- The input $\{a_i\}$ is an i.i.d. CM sequence
- The equalizer is infinite-length
- No additive noise is present

Assuming for simplicity that the constellation modulus equals 1, the transmitted symbols take the form

$$a_k = e^{j\phi_k} \ , \qquad (8)$$

and the CMA 1-2 cost function is given by

$$J(s) = E(|y_k| - 1)^2 \ \ (> 0) \ . \qquad (9)$$

The equilibria of the CMA 1-2 cost function are found by setting its first partial derivative w.r.t. each element of s equal to zero, which gives

$$E\left(1 - \frac{1}{|\sum_i a_{n-i}s_i|}\right) a_{n-k}^* \sum_i a_{n-i}s_i = 0, \ \forall \ k. \quad (10)$$

A class of settings that satisfy (10) are all settings $\{s_i\}$ that contain M non-zero elements of equal magnitude, α_M, $M = 1, 2, \ldots$: [2]

$$s_M = \alpha_M \ [\ \cdots \ 0 \ e^{j\theta_1} \ 0 \ \cdots \ 0 \ e^{j\theta_2} \ 0 \ \cdots \ 0 \ e^{j\theta_M} \ 0 \ \cdots] \ . \qquad (11)$$

The corresponding magnitudes α_M (see also [1]) are given by

$$\alpha_M = E\left(\frac{a_k^* \sum_{m=0}^{M-1} a_{k-m}}{|\sum_{m=0}^{M-1} a_{k-m}|}\right) \ . \qquad (12)$$

A simpler expression results if one sets the derivative of the cost function to 0 on a stationary point of the type of (11) w.r.t. α_M. This yields

$$\alpha_M = \frac{\gamma_M}{M} \ , \qquad (13)$$

where γ_M is defined as

$$\gamma_M = E|\sum_{i=1}^{M} a_i| \ . \qquad (14)$$

[2] Contrary to what is noted in [1], there may exist also other classes of stationary points

For $M = 1$, $\alpha_1 = 1$ and therefore $|y_k| = 1$, for all k. Consequently, $E(|y_k| - 1)^2 = 0$ in this case. As $E(|y_k| - 1)^2$ is in general a nonnegative quantity, we conclude that all the stationry points for $M = 1$ correspond to global minima of the CMA 1-2 cost function. These minima are given by $s = e^{j\theta}$ where θ is an arbitrary angle and all correspond to Zero-Forcing (ZF) equalizers. The fact that they rotate the constellation by a constant but unknown factor is a problem frequently encountered in BE and can be overcome by using differential encoding.

We now consider the case $M \geq 2$. The two stochastic quantities of the CMA 1-2 cost function are $E|y_k|^2$ and $E|y_k|$. Keeping (14) in mind and using the i.i.d. assumption, these quantities, evaluated at a stationary point of the form of (11) are given by

$$\begin{aligned} E|y_k|^2 &= M\alpha_M^2 \\ E|y_k| &= \alpha_M \gamma_M = M\alpha_M^2 \quad . \end{aligned} \tag{15}$$

Consider now the following perturbation of s_M

$$s_M^\epsilon = \sqrt{1+\epsilon}\,\alpha_M[\cdots e^{j\theta_1} \cdots e^{j\theta_2} \;\cdots e^{j\theta_M} \cdots] \;, \tag{16}$$

where ϵ is a very small positive number and define $y_k^+ = \sum_{i=-\infty}^{\infty} a_{k-i} s_{M,i}^\epsilon$, where $s_{M,i}^\epsilon$ denotes the i^{th} element of s_M^ϵ, $J^o = E(|y_k| - 1)^2$ and $J^+ = E(|y_k^+| - 1)^2$. Then we have

$$J^+ - J^o = M\;\alpha_M^2\;(2 + \epsilon - 2\sqrt{1+\epsilon}) > 0 \quad . \tag{17}$$

Therefore the following lemma holds:

<u>*Lemma 2.1:*</u> Under the above assumptions the CMA 1-2 has no local maxima points of the form (11).

We are now interested in the existence of local minima of the cost function (9). The second derivative of $J(y)$ around a stationary point of the form (11) can be found to be given by

$$\frac{\partial^2 J}{\partial s_k^* \partial s_l} = \begin{cases} 1 - \frac{1}{2\alpha_M} E \frac{a_k^* a_l}{|\sum_{i=1}^M a_i|}, & k, l \in \{1, \ldots, M\} \\ \delta_{kl}\;, & else \;\;, \end{cases} \tag{18}$$

and therefore the Hessian matrix ($M \times M$ since elsewhere it is 0) can be written as

$$H_M = I_M - \frac{1}{2\alpha_M} E \frac{1}{|\sum_{i=1}^M a_i|} \begin{bmatrix} a_1^* \\ \vdots \\ a_M^* \end{bmatrix} \begin{bmatrix} a_1^* \\ \vdots \\ a_M^* \end{bmatrix}^H \quad . \tag{19}$$

The eigenvalues of this matrix can be found to be

$$\begin{cases} \lambda_1 = \frac{1}{2} \\ \lambda_2 = \frac{M - \frac{1}{2}}{M-1} - \frac{M^2}{2(M-1)} \frac{1}{E|\sum_{i=1}^M a_i|} E \frac{1}{|\sum_{i=1}^M a_i|} \end{cases} \quad . \tag{20}$$

It can be shown that $\lambda_2 < 1$. We now examine separately the two following cases:

- M is even

In this case, for any symmetrical CM constellation this means that $\lambda_2 = -\infty$. As λ_1 is positive, it turns out that the stationary points of the form (11) for M even are neither minima, nor maxima, they are therefore saddle points. We have therefore the following lemma:

<u>*Lemma 2.2:*</u> All stationary points of the form (11) for M even are saddle points of the cost function (9).

- M is odd

Evaluating the expression of λ_2 for CM constellations, it turns out that for some of them there exist values of M (especially $M = 3$) for which $\lambda_2 > 0$. This means that these settings correspond to local minima (as λ_1 is also positive). Therefore the following lemma holds:

<u>*Lemma 2.3:*</u> Some of the stationary points of the form (11) for M odd are local minima of the cost function (9).

However, as the number of constellation points on the unit circle increases, the eigenvalues λ_2 tend quickly towards negative values, and therefore the corresponding settings correspond all to saddle points. The same holds of course in the asymptotic case of a CM input distribution whose angle is uniformly distributed in $[0, 2\pi)$. But in this case the CMA 1-2 coincides with the DD algorithm, as already mentioned. We therefore have the following result:

<u>*Lemma 2.4:*</u> When the input signal is CM and its angle is continuously-uniformly distributed in $[0, 2\pi)$, then all equilibria of the CMA 1-2 (or the DD) cost function of the form (11) are saddle points, except for $M = 1$.

The above lemma constitutes our main result and is indicative of the good performance of DD equalizers when used for CM signals.

3 The non-CM case

In the case of non-CM constellations DD equalization usually does not have a satisfactory performance, especially when the channel eye is initially closed. We consider for simplicity the PAM case. The DD cost function in this case is

$$J^D(s) = E(y_k - dec(y_k))^2 \;\;, \tag{21}$$

and the stationary points are given by the equation

$$\begin{aligned} & E(\sum_i a_{n-i}s_i - dec(\sum_i a_{n-i}s_i))a_{n-k} \;\times \\ & (1 - \delta(\sum_i a_{n-i}s_i - bis(\sum_i a_{n-i}s_i))) \;= 0 \;\;, \end{aligned} \tag{22}$$

where $bis(y)$ is the closest point to y that belongs on a bisector between the different PAM levels, and

$\delta(\cdot)$ is the Dirac function. Neglecting singularities and assuming that s has M non-zero taps ($s = (\cdots 0\ s_{k_1}\ 0 \cdots 0\ s_{k_M}\ 0 \cdots)$), the non-zero elements of s should satisfy

$$s_{k_i} = E(a_{n-k_M} dec(\sum_{j=1}^{M} s_{k_j} a_{n-k_j})) \ . \qquad (23)$$

If now we constrain the solutions to be of equal magnitude α_M, the solution of (23) is no longer unique (as for the Sato or the CMA 1-2) for each M, and moreover, cannot be easily expressed analytically. However it is possible in each case to find numerically the different amplitudes that satisfy (23). For example, in a 4-PAM case with $M = 2$ there exist 5 different stationary points for the DD cost function, as compared to a unique stationary point for the Sato cost function. This can be seen in figure 3, where we have plotted the difference of the two terms in (23) and the corresponding equation for Sato, respectively. Moreover, it

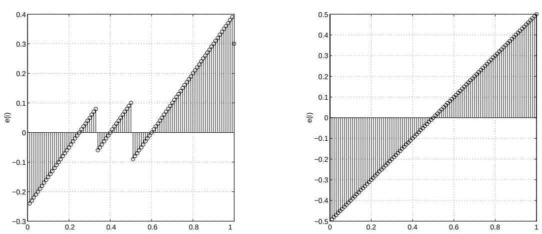

Figure 3: 4-PAM: The stationary points on the line $s_1 = s_2$ for DD and Sato, respectively

can be proven that 2 of these solutions are local maxima (compared to the absence of local maxima for the Sato or the CMA 1-2). This multitude of stationary points that can be either local minima, saddle points, or local maxima is responsible for the very different shape of the DD cost function in a multi-level PAM system, as compared to the Sato (or CMA 1-2) cost function. This can be seen in figure 4, where the two corresponding cost functions have been plotted for a 4-PAM system and $M = 2$. Note in this figure the fact that in the Sato cost function, the regions separating local minima are open cone-shaped regions, whereas the corresponding regions of the DD cost function are closed, and various local maxima and saddle points separate the local maxima. This fact has also been discussed in [6]. This is why escaping from a local minimum is much more difficult in the DD case (for non-CM constellations).

4 A CMA-DD hybrid corresponding to a novel Generalized-Sato scheme

The results of the previous section suggest that the principal factor that causes the misbehaviour of the DD algorithm is the multiple constellation amplitudes, whereas in the case of a single constellation amplitude its performance should be similar to the one of the CMA 1-2 algorithm. Based on this fact, we propose the following CMA-DD hybrid method for equalization of non-CM signals:

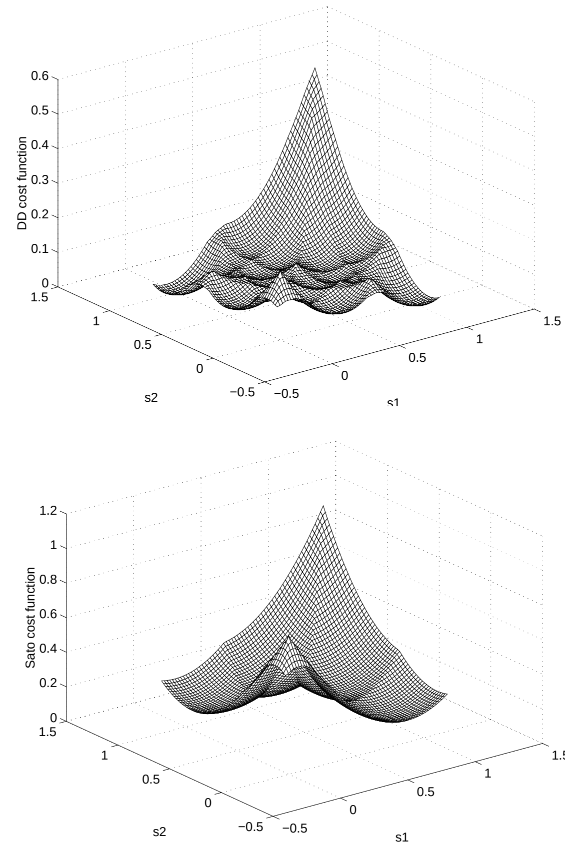

Figure 4: 4-PAM: the DD and Sato cost functions ($M = 2$), respectively

- Construct from the non-CM constellation a CM constellation by associating each symbol a_i with the complex number $r_1 sign(a_i)$

- Use the CM constellation to make decisions about the received samples.

An example of this principle can be seen in figure 5, where the non-CM constellation is 16-QAM ('+' denotes a constellation symbol, '$*$' a projected symbol). The LMS-like algorithm corresponding to this principle is a novel GSATO-like algorithm in the sense that it respects Sato's philosophy better than the already known GSATO. Namely, our principle preserves the constant modulus character of the constellation, but not in the reduced-constellation sense of the GSATO. A short discussion about this principle can also be found in [7] where different algorithms for the implementation of this principle are given.

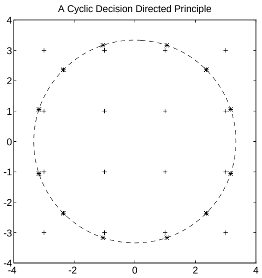

Figure 5: A CMA-DD hybrid principle: 16-QAM

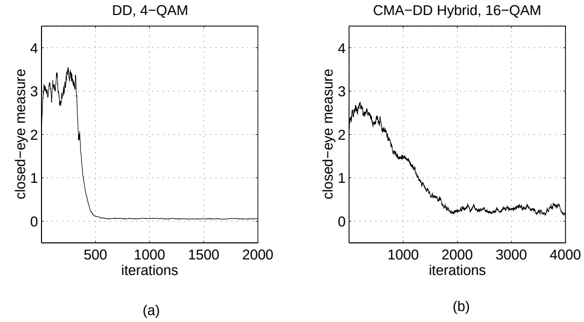

Figure 6: The opening of the channel eye achieved by a DD (a) and a CMA-DD Hybrid (b) algorithm

5 Computer simulations

Some computer simulations that verify our theoretical findings can be shown in figure 6. Figure (6a) shows the opening of the channel eye achieved by a DD algorithm in a 4-QAM case and verifies the fact that eye opening can be achieved with a DD equalizer in the CM-case. The channel is FIR with impulse response $c = [1\ -3\ 3\ 2]$, the equalizer has 11 taps and is initialized with a unique non-zero middle tap equal to 1, the SNR is 30 dB and the algorithm used is a Normalized Sliding Window DD (NSWDD) algorithm (see [7]) with parameters $\bar{\mu} = .1, L = 2$. Figure (6b) shows the performance of an equalizer that implements the CMA-DD hybrid principle introduced above, for the same channel. Again we use a NSW algorithm, with parameters $\bar{\mu} = .05, L = 2$. The opening of the initially closed channel eye in this case too is indicative of the performance of this BE scheme.

6 Conclusions

We have taken a close look at the problem of DD BE in the particular case of CM signals. An analysis of the stationary points of the CMA 1-2 cost function has shown the absence of local minima for the DD cost function in the asymptotic case of a CM distribution whose angle is uniformly distributed in $[0, 2\pi)$. This result parallels the one presented in [4], where it was proven that the Sato algorithm has no local minima in the case of a continuous sub-Gaussian input distribution and justifies the following statement: DD equalizers are capable of opening an initially closed channel eye when the input signal is CM. Combined with the fact that DD equalizers have a reduced steady-state error as compared to other blind equalizers, this result indicates (in contrast to what is widely believed to date) that DD equalization is a valid BE method for CM constellations.

Our stationary-point analysis also provides useful insight into the performance of DD equalizers in the non-CM case: clearly, it is the existence of multiple constellation amplitudes that is responsible for the bad shape of the cost function in this case. Based on this remark, we propose a CMA-DD hybrid scheme for BE which constitutes a novel Generalized Sato-like scheme for QAM constellations that respects Sato's philosophy more closely than the already existing GSATO algorithm: a *projected* constellation on a circle of radius r_1 is used (instead of a *reduced* constellation) in order to form the decisions about the transmitted symbols. The performance of algorithms based on this scheme as well as the ability of DD equalizers to open an initially closed channel eye are verified by our computer simulations.

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