Models of Blocking Probability in All-Optical Networks with and without Wavelength Changers

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Abstract

We introduce a traffic model for circuit switched all-optical networks (AONs) which we then use to calculate the blocking probability along a path for networks with and without wavelength changers.

We investigate the effects of path length, switch size, and interference length (the expected number of hops shared by two sessions which share at least one hop) on blocking probability and the ability of wavelength changers to improve performance.

Our model correctly predicts unobvious qualitative behavior demonstrated in simulations by other authors.

Keywords: All-Optical Networks, Wavelength Changers, Traffic Models

1 Introduction

In an All-Optical Network (AON), the signals remain in the optical domain from the origin to the destination. We consider AONs supporting point-to-point sessions between users. Without wavelength changers, a session must use the same wavelength on every fiber. With wavelength changers, a session may use a different wavelength on each fiber it occupies. In either case, two sessions simultaneously using the same fiber must be on different wavelengths.

In this paper, we model the probability of a path being blocked in circuit switched all-optical networks (AONs) with and without wavelength changers.

Consider Fig. 1. Suppose that user A requests a session to user B over some path of an AON and that there are H hops (fibers) from A to B on this path (we do not count the access or exit fibers). We consider networks where each session requires a full wavelength of bandwidth and there are F available wavelengths.

We assume that A and B are not currently active at the time of the session request. Therefore, there are no busy wavelengths on the access or exit fiber, and in particular a session cannot enter the requested path at node $H+1$. However, sessions may enter or exit the path at each of the first H intermediate nodes provided that no two sessions on the same fiber use the same wavelength. Any session which uses at least one of the H fibers on any wavelength is termed an *interfering session*.

With wavelength changers, this is a conventional circuit switched network. In this case, the request between A and B is blocked only if one of the H fibers is *full*, (a fiber is full when it is supporting F sessions on different wavelengths).

Now consider a network without wavelength changers. Since the session must be assigned the same wavelength to use on each fiber, the session can be *honored* on this path only if there exists a *free* wavelength, i.e. a wavelength which is unused on each of the H fibers. Therefore, there is the possibility in such networks that requests will be blocked even if all links are supporting less than F sessions. For instance, suppose that $H = F$ and wavelength i is used on hop i only. Then each fiber along this path has only one active session but there is no wavelength available to the request.

Obviously a network with wavelength changers is more flexible and has a smaller blocking probability; however, quantitative results on the usefulness of wavelength changers have been mixed. The most wavelength efficient topologies currently known do not require wavelength changers, and these topologies are nearly optimal in the sense that they use almost the minimum number of wavelengths [1, 2]. However, these networks have carefully designed topologies which are unlikely to be implemented on a national scale. On the other hand, simulations of random topologies have indicated a modest benefit of wavelength changers [3]. In the other extreme, examples can be constructed for which wavelength changers provide a very large performance gain [2].

We propose a traffic model which we use to calculate blocking probability along a path with and without wavelength changers. We use this model to investigate the performance gain, measured in terms of the increase in fiber utilization, achieved by the use of wavelength changers. Also, we investigate the effects of some topological and routing properties on this gain.

The general form of the model is presented in Section 4. First, two interesting special cases are developed in Sections 2 and 3.

The first form of the model, presented in Section 2, is based on Lee's well known traffic model for circuit switched networks [4]. Although simple, several interesting qualitative conclusions can be drawn about the effects of path length on blocking probability. We find that path length is a key design parameter for networks without wavelength changers. This conclusion has previously been pointed out by a variety of authors in different ways, e.g. by simulations in [3, 5]

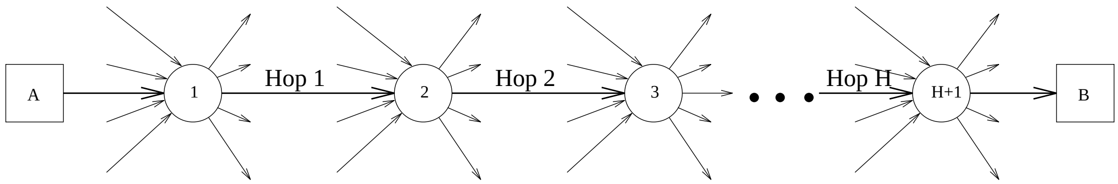

Figure 1: **An H hop request. Requested links are shown in bold. The other links are interfering links. We assume no blocking on the link between A and node 1 and between node $H+1$ and B.**

and by a theorem and an example in [2]. In order to keep the blocking probability small, path length must be kept small since it becomes less likely to find a free wavelength on all the hops of a path as the number of hops increases. That is, the number of interfering sessions on a path tends to increase with the number of hops.

In Section 3, we extend the above model by using Pippenger's improvement [6] to Lee's model. We predict effects of path length and nodal degree (switch size) on blocking probability. We will see that the switch size is important because networks with large switches tend to mix sessions more than networks with small switches. That is, the number of interfering sessions on a path increases with switch size as well as the number of hops. Although the effects are secondary to path length, we show that they are significant in networks without wavelength changers if the switch size is small. In addition, we compare our results to simulations performed by Sivarajan and Ramaswami in [5].

A third important parameter, *interference length* L, is a function of both the network topology and routing algorithm. Loosely speaking, the interference length is the number of hops shared by two sessions. The effects of L are captured by the most general version of our model which is presented in Section 4. We show that networks with large interference length reduce the need for wavelength changers since the number of interfering sessions tends to decrease as the interference length increases.

Finally in Section 5, we summarize our conclusions.

2 The Effects of Path Length

We start by making the standard series independent link assumption introduced by Lee and commonly used in the analysis of circuit switched networks [4, 7]. In particular, we assume that in steady state, a request sees a network where a wavelength is used on a hop statistically independently of other hops and other wavelengths. Lee's model tends to overestimate the blocking probability in circuit switched networks [7] and it would be surprising if that were not the case in AONs.

Let ρ be the probability that a wavelength is used on a hop. Note that since ρF is the expected number of busy wavelengths, ρ is a measure of the fiber utilization along this path.

First consider networks with wavelength changers. The probability P_b' that the session request between A and B is blocked is the probability that there exists a hop with all wavelengths used, i.e.

$$P_b' = 1 - (1-\rho^F)^H \tag{1}$$

Let q be the achievable utilization for a given blocking probability in networks with wavelength changers i.e.

$$q \stackrel{\mathrm{def}}{=} \left[1-(1-P_b')^{1/H}\right]^{1/F} \tag{2}$$

$$\approx \left(\frac{P_b'}{H}\right)^{1/F} \tag{3}$$

where the approximation is valid for small P_b'/H.

In Fig. 2, we plot the achievable utilization q for $P_b = 10^{-3}$. The utilization is plotted as a function of the number of wavelengths for $H = 5, 10, 20$ hops. Notice that the effect of path length on utilization is small. It is apparent that a good routing policy would minimize the congestion of links at the expense of more hops if possible, although minimum hop routing can be a good heuristic to minimize congestion by reducing the total bandwidth consumed by a single session. Notice also that q rapidly approaches 1 as $F \to \infty$. This is a demonstration of the well known fact that large trunk groups are more efficient than small trunk groups.

Now consider a network without wavelength changers. Again let ρ be the probability that a wavelength is used on a link. In the absence of wavelength changers, the probability of blocking P_b is the probability that each wavelength is used on at least one of the H hops, i.e.

$$P_b = [1-(1-\rho)^H]^F \tag{4}$$

Now let p be the achievable utilization for a given blocking probability in networks without wavelength changers, i.e.

$$p \stackrel{\mathrm{def}}{=} 1-\left(1-P_b^{1/F}\right)^{1/H} \tag{5}$$

$$\approx -\frac{1}{H}\ln\left(1-P_b^{1/F}\right) \tag{6}$$

where the approximation is valid for large H and $P_b^{1/F}$ not too close to 1. To see this, notice that $(1-x)^\alpha \approx 1+\alpha\ln(1-x)$ as long as $\alpha\ln(1-x)$ is small. Note that the achievable utilization is inversely proportional to H.

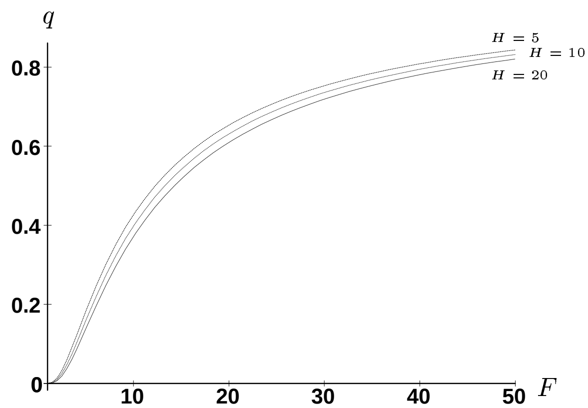

Figure 2: **Wavelength utilization increases with the number of wavelengths and the effects of H are small in networks with wavelength changers.** $P_b = 10^{-3}$

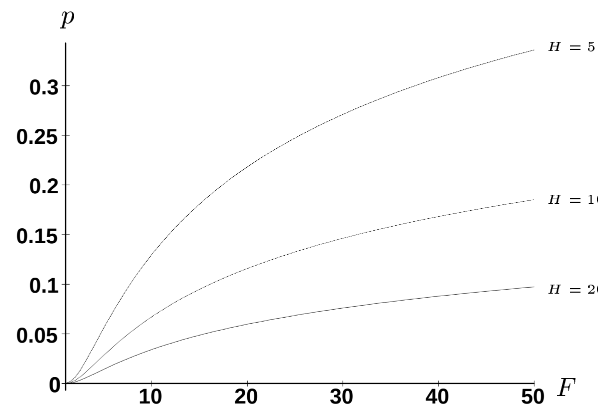

Figure 3: **Wavelength utilization increases with the number of wavelengths and the effects of H are significant in networks without wavelength changers.** $P_b = 10^{-3}$

In Fig. 3, we plot the achievable utilization p for $P_b = 10^{-3}$. The utilization is plotted as a function of the number of wavelengths for $H = 5, 10, 20$ hops. Notice that unlike the previous case, the effect of path length is dramatic.

It seems apparent that the diameter of a network without wavelength changers should be kept small, else fibers will be greatly underutilized. It is also apparent that a good routing policy for networks without wavelength changers consider path lengths in hops, as well as the congestion of links.[1] Notice that the main reason for minimizing hops in networks without wavelength changers is different than the reason for minimizing hops in circuit switched networks or networks with wavelength changers. In the former case, we are trying to reduce the expected number of interfering sessions on a path, whereas in the latter case, we are trying to minimize congestion on the links.

Notice also that p approaches 1 as $F \to \infty$, i.e. the well known adage that large trunk groups are more efficient than small trunk groups continues to hold for networks without wavelength changers. This effect was also observed analytically and using simulations in [3]. However the convergence is so slow that this effect may be irrelevant for practical systems where the number of wavelengths is limited.

As a measure of the benefit of wavelength changers, define the *gain* $G = q/p$ as the increase in utilization for the same blocking probability. Setting $P_b = P_b'$ and solving for q/p, we get

$$G \stackrel{\text{def}}{=} \frac{q}{p} \quad = \quad \frac{\left[1-(1-P_b)^{1/H}\right]^{1/F}}{1-\left(1-P_b^{1/F}\right)^{1/H}} \tag{7}$$

[1] However we will show in the next two sections that this model overestimates this effect.

$$\approx \quad H^{1-\frac{1}{F}} \frac{P_b^{1/F}}{-\ln\left(1-P_b^{1/F}\right)} \tag{8}$$

for the wavelength changing gain. This gain comes at the cost of increased hardware. The approximation is valid for small P_b, large H, and moderate F (so that $P_b^{1/F}$ is not too close to 1).

Typical plots of G versus F are shown in Figs. 4a-c. In each figure, G is shown as a function of F for 5, 10, and 20 hops. Fig. 4a shows G for a blocking probability $P_b = 10^{-3}$. Likewise Figs. 4b and 4c show G for $P_b = 10^{-4}$ and $P_b = 10^{-5}$, respectively. The gain increases as the blocking probability decreases; however the effect is small as long as P_b is small.

Notice that $G = 1$ if either $H = 1$ or $F = 1$ since in either of these cases there is no difference between a system with or without wavelength changers. So for instance, wavelength changers are useless in two-stage (1 hop) switching networks.

As F increases, the gain increases until G peaks somewhere near $F \approx 10$ ($q \approx .5$) for all cases shown. As can be seen from the figures, the maximum gain is close to $H/2$. After peaking, the gain slowly decreases and approaches 1 as $F \to \infty$ for the simple reason that large trunk groups are more efficient, i.e. both p and q approach 1. The convergence is extremely slow since the convergence of p is extremely slow.

It's interesting to note that even for a moderate number of wavelengths, we are operating in a regime where there is diminishing returns for the use of wavelength changers. That is, as we increase the number of wavelengths, the node complexity increases and the benefit of the hardware decreases.

Now consider G as a function of the number of hops H. Notice that for large F, the gain is roughly linear in the number of hops, basically because q is nearly independent of H and p is inversely proportional to

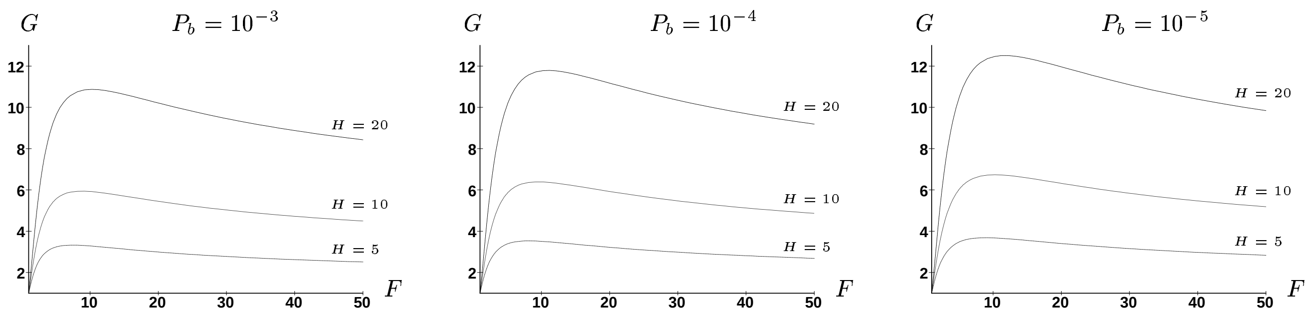

Figure 4: **Wavelength Changing Gain. The effects of H are large.**

H. It can be shown that G is never more than

$$G \leq H^{1-\frac{1}{F}}. \tag{9}$$

Therefore interestingly, for a two wavelength system, G grows more slowly than $\sqrt{H}$.

In summary, for a moderate to large number of wavelengths, the benefits of wavelength changers increase with the number of hops and decrease with the number of wavelengths. The benefits also increase as the blocking probability decreases; however the effect is small as long as P_b is small.

We argue in the next two sections that we have overestimated the gain in efficiency that wavelength changers provide.

3 The Effects of Switch Size

The model in the last section correctly identifies hop length as a major design criteria. However that simplified approach does not identify another important parameter: switch size. As stated previously, large switches tend to mix signals more than small switches. In this section we account for this effect. We argue that Lee's model overestimates the gain in efficiency that wavelength changers provide.

Define the H links connecting A to B to be the *requested links* and the links entering or exiting the intermediate nodes the *interfering links*. Any call using a requested link is termed *interfering*. We assume for simplicity that each node has Δ incoming and Δ outgoing unidirectional fibers, including the fibers on the path. A node without wavelength changers can be modeled as Δ frequency demultiplexers followed by a $\Delta \times \Delta$ switch per wavelength, followed by Δ frequency multiplexers. All wavelengths can be switched independently. In networks with wavelength changers, the nodes also contain wavelength changing devices before the demultiplexers and after the multiplexers. Each wavelength changer can be in one of $F!$ states.[2]

We assume that a wavelength is used on an incoming interfering link with probability ρ. Furthermore, we assume that all incoming interfering links are independent and that different wavelengths are independent. We also assume that the switches are equally likely to be in any of the $(\Delta!)^F$ possible settings (each switch can be set to one of $\Delta!$ settings on each wavelength). For networks with changers, we assume that each changer is set statistically independently to one of $F!$ states.

Notice that the probability that a wavelength λ is used on an interfering link is not the probability that λ is used on a requested link. The former is by definition ρ. To calculate the latter probability ρ_i, notice that because the access link is assumed empty at the time of the request, λ is not used on hop i if the first i switches connect hop 0 to hop i on λ. This occurs with probability Δ^{-i}. If hop i is not connected to hop 0 on λ, then hop i is connected to one of the interfering links on λ. In this case, λ is used on hop i with probability ρ. Therefore,

$$\rho_i = \rho(1 - \Delta^{-i}) \tag{10}$$

Notice that ρ_i increases to ρ fairly rapidly as we move down the chain. Also, the average utilization along the chain $H^{-1}\sum_{i=1}^{H} \rho_i \approx \rho$ if $\Delta^H >> 1$. For these reasons, we will continue to call ρ the utilization.

Consider first the blocking probability in a network without wavelength changers. Suppose that wavelength λ is free on requested hop $i-1$. If the switch for λ is set such that link $i-1$ is connected to link i, then λ is not used on link i. The probability of this is $\frac{1}{\Delta}$. Otherwise, link i is fed by one of the interfering links on λ. In this case, λ will not be used on i with probability $(1-\rho)$. Therefore,

$Pr\{\lambda \text{ free on hop } i \mid \lambda \text{ free on hop } i-1\} =$

$$\begin{aligned} \frac{1}{\Delta} + \left(1 - \frac{1}{\Delta}\right)(1-\rho) &= 1 - \left(1 - \frac{1}{\Delta}\right)\rho \qquad (11) \\ &> 1 - \rho \end{aligned}$$

Now since all wavelengths and all incoming interfering links are assumed to be independent, the blocking

[2] This design is rearrangeably non-blocking. Strict sense non-blocking switches can be designed but their introduction would only complicate the discussion here.

probability is easily calculated to be

$$\begin{aligned} P_b = & \ [1 - Pr\{\lambda \text{ free on all hops}\}]^F \\ = & \left[1 - \prod_{i=1}^{H} Pr\{\lambda \text{ free on hop } i \mid \lambda \text{ free on } i-1\ \}\right]^F \\ = & \left(1 - \left[1 - \left(1 - \frac{1}{\Delta}\right)\rho\right]^H\right)^F \end{aligned} \quad (12)$$

where hop 0 is considered to be the fiber leaving user A entering the first node, and we have assumed that user A does not have any other active calls. Notice that we have also used the fact that the switches are set independently on each hop.

Notice that large Δ (more mixing) degrades performance. In one extreme $\Delta = 1$, and there are no interfering links. In this case the H hops look like 1 hop since all calls enter at node 1 and leave at node H. In the other extreme $\Delta \to \infty$, and the event λ used on i becomes independent of the event λ used on $i-1$. This is so because with very high probability, the switch is set such that $i-1$ is not connected to i on λ.[3]

Inverting eqn. (12) gives the achievable utilization for a network without wavelength changers. Denoting this utilization by p,

$$p = \frac{\Delta}{\Delta - 1}\left[1 - \left(1 - P_b^{1/F}\right)^{1/H}\right] \quad (13)$$

Comparison with eqn. (4) shows that p can be $\Delta/(\Delta-1)$ larger than predicted by the simplified model (the $\Delta = \infty$ model). We shall see shortly that the same conclusion does **not** hold for networks with wavelength changers, i.e. the effect of Δ is much smaller in this case. Therefore finite Δ reduces the wavelength changing gain by about $(\Delta-1)/\Delta$. Since we observed earlier that $G \lesssim H/2$ for many situations of interest, the benefits of using wavelength changers in networks with small diameter D and small Δ are limited.

Before deriving the blocking probability for networks with wavelength changers, we compare our model to simulation results presented in [5]. We will see that our model makes unobvious qualitative predictions confirmed by those simulations. However, more simulations and experience are required for the model to be fully evaluated.

Sivarajan and Ramaswami considered DeBruijn graphs for store and forward as well as circuit switched AONs. We discuss their results on circuit switched AONs here. They considered directed DeBruijn graphs where each node has in- and out-degree Δ (a small number of nodes have self-loops). The number of nodes is $N = \Delta^D$ where D is the diameter. The authors considered two possible designs for $N = 1024$.

[3] If the carried load between A and B is more than one call, then this assumption is violated. In this case, the effect of mixing is diminished since there are more calls from A to B and less interfering calls.

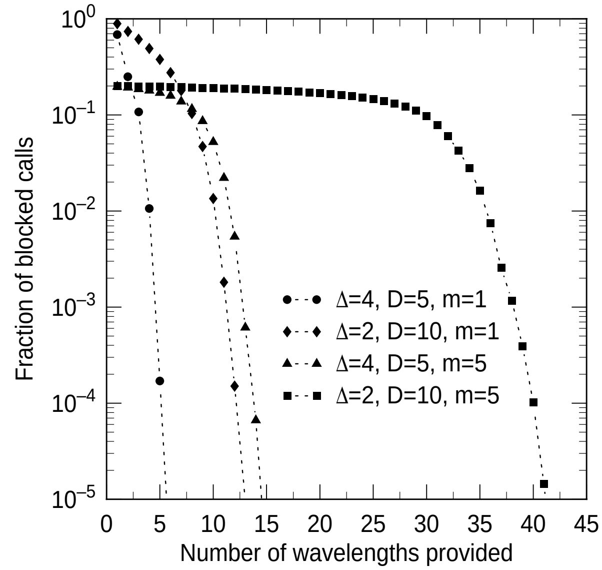

Figure 5: **Results of Sivarajan and Ramaswami [5]. Reproduced with the authors' permission.**

The first network had $\Delta = 4$ and $D = 5$. The second had $\Delta = 2$ and $D = 10$. For each design, they simulated the DeBruijn network under the assumption that each node has $m = 1$ and $m = 5$ duplex session requests, i.e. if A is talking to B then B is talking to A and the session request is blocked if either direction of the request cannot be honored. The requests were handled sequentially, e.g. the first request cannot be blocked because the network is empty (in fact, the first F requests cannot be blocked). Their results are shown in Fig. 5.

It is certainly not surprising that the 10 diameter networks require more wavelengths than the 5 diameter networks under the same load (see eqn. (4)). However it is certainly not a priori obvious that the 5 diameter network with 5 calls per node requires more wavelengths than the 10 diameter network with 1 call per node.

Using eqn. (12), we calculate the steady state blocking probability as a function of the number of wavelengths. We set $\rho F \leq F$ to be the average number of sessions per link under the assumption that all calls are honored, i.e.

$$\rho = \min\{1, \frac{2mN\bar{H}}{N\Delta F}\} = \min\{1, \frac{2m\bar{H}}{\Delta F}\} \quad (14)$$

where $2mN$ is the number of one-way session requests, $\bar{H}$ is the average number of hops used by a call in each direction and $N\Delta$ is the total number of links in the network.[4] For the blocking probability, we use eqn. (12) averaged over the number of hops, i.e.

$$P_b \approx \frac{2}{N}\sum_{H=1}^{D}(\Delta-1)\,\Delta^{H-1}\left(1 - [1 - \left(1 - \frac{1}{\Delta}\right)\rho]^H\right)^F$$

[4] A better model for large P_b would be to set $\rho = 2m\bar{H}(1 - P_b)/\Delta F$.

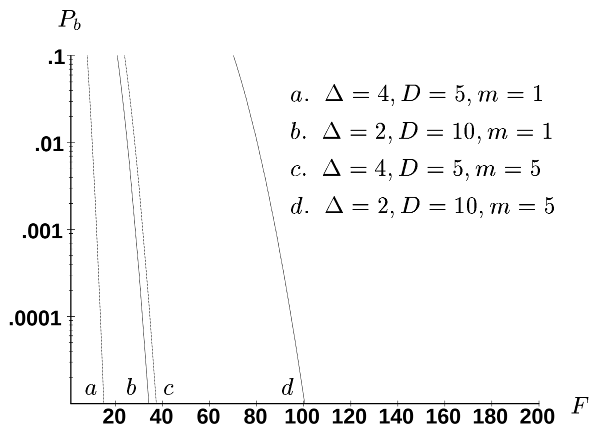

Figure 6: **Analytic Approximation of P_b .**

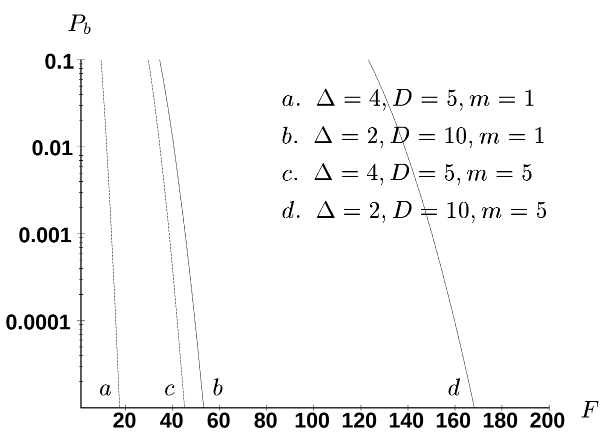

Figure 7: **Analytic Approximation of P_b using $\Delta = \infty$.**

since a node can reach about $(\Delta-1)\Delta^{H-1}$ nodes in H hops and there are $N = \Delta^D$ nodes. The factor of 2 in the numerator accounts for the fact that a duplex call request is blocked if either of the one way requests are blocked, ignoring the $o(P_b^2)$ term.

Our results are shown in Fig. 6. Notice that we identify the same ordering in terms of required number of wavelengths as the simulations; however our blocking probabilities are larger. Likely reasons for this will be discussed momentarily. First, consider Fig. 7 which shows the blocking probability calculated using the independent requested link assumption (eqn. (4)). Notice that it predicts even higher blocking probabilities and incorrectly orders the designs. In particular, the 5 diameter with 5 calls per node (line c) requires **less** wavelengths than the 10 diameter network with 1 call per node (line b), a conclusion not supported by the simulations. We therefore conclude that the shorter diameter network requires more wavelengths because it has a higher load ($m = 5$ versus $m = 1$) **and** because it has larger switches and therefore more mixing.

One reason our model predicts higher P_b than the simulations is that we have calculated a steady state blocking probability, whereas in [5], the calls were set up sequentially and the network was not allowed to progress to steady state. In particular, the calls were routed in m rounds, where each round tried to route a random set of duplex calls with at most one call per user per round. Since the calls were set up sequentially, the first call sees an empty network, the second call sees 1 active call, etc. Therefore, we expect their results are an underestimate of the steady state blocking probability. We expect our model overestimates P_b and that the true answer lies in between.[5]

Another likely reason our curves lie above the simulations is the wavelength assignment scheme used by Sivarajan and Ramaswami. In particular, if a path has more than one free wavelength available, the lowest numbered wavelength is used. This has the effect of making the load on each wavelength different. We speculate that this reduces the blocking probability.

Now we calculate the effect of mixing on networks with wavelength changers. In this case we will see that the effects are very small except for some special cases. Let f_i be the event that link i is full, i.e. all wavelengths are being used. Also let $\bar{f}_i$ be the event that link i has at least one available wavelength. Then

$$\begin{aligned} P_b' &= 1 - Pr\{\bar{f}_1, \bar{f}_2, \bar{f}_3, ..., \bar{f}_H\} \\ &= 1 - \prod_{i=1}^{H} Pr\{\bar{f}_i \mid \bar{f}_{i-1}\} \end{aligned} \tag{15}$$

where the second equality follows from the fact that switch settings are independent.

Now

$$\begin{aligned} Pr\left(f_i\right) &= Pr\left(f_i \mid f_{i-1}\right) Pr\left(f_{i-1}\right) \\ &\quad + Pr\left(f_i \mid \bar{f}_{i-1}\right) Pr\left(\bar{f}_{i-1}\right) \end{aligned}$$

and $Pr\left(f_i \mid \bar{f}_{i-1}\right) = 1 - Pr\left(\bar{f}_i \mid \bar{f}_{i-1}\right)$. Therefore,

$$Pr\left(\bar{f}_i \mid \bar{f}_{i-1}\right) = 1 - \frac{Pr\left(f_i\right) - Pr\left(f_i \mid f_{i-1}\right) Pr\left(f_{i-1}\right)}{1 - Pr\left(f_{i-1}\right)}$$

The calculation of $Pr\left(f_i \mid f_{i-1}\right)$ is complicated slightly because of the wavelength changers. Without wavelength changers,

$$\begin{aligned} Pr\left(f_i \mid f_{i-1}\right) &= \left(Pr\{\lambda \text{ used on } i \mid \lambda \text{ used on } i-1\}\right)^F \\ &= \left(\frac{1}{\Delta} + \rho\left(1 - \frac{1}{\Delta}\right)\right)^F \end{aligned} \tag{16}$$

[5]We used our model to approximate a sequential P_b in a variety of ways. Each method lowered the blocking probabilities, and in each case, the proper ordering of the 4 curves was preserved.

The second equality is derived in the same way eqn (11) was derived. With wavelength changers, condition on the states of the changers. A little thought should convince the reader that the answer does not change.

Finally, since all wavelengths are independent, $Pr(f_i) = \rho_i^F$ where recall that $\rho_i = \rho(1-\Delta^{-i})$ is the probability that a wavelength is used on hop i. The path blocking probability is therefore

$$P_b' = 1 - \prod_{i=1}^{H} \left[1 - \frac{\rho_i^F - \rho_{i-1}^F \left(\frac{1}{\Delta} + \rho\left(1 - \frac{1}{\Delta}\right)\right)^F}{1-\rho_{i-1}^F} \right] \quad (17)$$

where

$$\begin{aligned} \rho_0 &= 0 \\ \rho_i &= \rho\left(1-\Delta^{-i}\right) \qquad i = 1, 2, ..., H. \end{aligned}$$

in a network with wavelength changers.

The effects of Δ on P_b' are small unless Δ, H and F are small. Intuitively, in networks with wavelength changers the blocking probability is roughly the probability that the heaviest loaded link is full, i.e. $P_b' \approx \rho_H^F$. To see this, notice that P_b' is at least this probability and no more than $P_b' \leq \sum_{i=1}^H \rho_i^F \leq H\rho_H^F$ since the ρ_i's are increasing with i. Therefore, the achievable utilization q is approximately

$$\left(\frac{\Delta^H}{\Delta^H - 1}\right)\left(\frac{P_b'}{H}\right)^{1/F} \leq q \leq \left(\frac{\Delta^H}{\Delta^H - 1}\right)(P_b')^{1/F}$$

So if $H^{1/F}$ is small and Δ^H is large, $q \approx P_b^{1/F}$.

Since the effects of Δ are small, a good approximation for q is obtained by letting $\Delta \to \infty$ and inverting eqn. (17). In that case, eqn. (17) becomes eqn. (1) and

$$q \approx \left(1 - (1-P_b')^{1/H}\right)^{1/F} \quad (18)$$

Even for $\Delta = 2$, this approximation works very well for large H but slightly underestimates q for paths of moderate length. For instance, for $\Delta = 2$ and $F \geq 5$, the approximation is within $15\%, 7\%$ and 5% of the exact values for $H = 5, 10,$ and 20 respectively. For $F \leq 5$, the error is $20-25\%$.

Now, define the gain to be the ratio of the average utilization along the path with wavelength changers versus without changers.

$$\begin{aligned} G &= \frac{\frac{1}{H}\sum_{i=1}^H (1-\Delta^{-i})q}{\frac{1}{H}\sum_{i=1}^H (1-\Delta^{-i})p} = \frac{q}{p} \\ &\approx \left(\frac{\Delta - 1}{\Delta}\right) \frac{\left[1-(1-P_b)^{1/H}\right]^{1/F}}{1-\left(1-P_b^{1/F}\right)^{1/H}} \end{aligned} \quad (19)$$

which is $(\Delta - 1)/\Delta$ smaller than eqn. (8).

In summary, for a moderate to large number of wavelengths, the benefit of wavelength changers increases with the number of hops and the nodal degree while decreasing with the number of wavelengths. Since the diameter D of a network tends to decrease with increasing switch size, there is a topological design trade-off between Δ and D.

We expect the above model to accurately predict the qualitative behavior of regular networks such the DeBruijn Networks. We also also expect the model to work fairly well for random graphs.[6] However, as will be discussed in the next section, the model does not accurately predict the behavior of networks with special topologies.

4 The Effects of Interference Length

In this section, we present the general form of the model. First, consider the following two motivating examples. Aggarwal, et. al., reported a network with nodal degree 2 and $O(N)$ hops, where N is the number of users [2]. This network had a lot of mixing; in particular, each call interfered with $O(N)$ other calls. The network required 2 wavelengths with wavelength changers and N without changers. These results are consistent with the analysis of the previous sections, i.e. wavelength changers help a lot because of the large number of hops. Now consider a unidirectional ring network. The nodal degree is 2, the average hop length is $O(N)$, and a call interferes with $O(N)$ other calls. However this network requires $O(N)$ wavelengths with or without wavelength changers, i.e. wavelength changers can only reduce the required number of wavelengths by a constant factor for any N. This result is inconsistent with the previous model.

The difference between the two networks is that in [2], two sessions which share some link, share exactly 1 link. On the other hand, in a ring network, the *interference length* (the expected number of links shared by two sessions which share some link) is $O(N)$. We will see that as the interference length increases, the relative benefit of wavelength changers decreases. In the model of Section 3, the assumption that each switch is set to an arbitrary state on each wavelength means that an interfering call stays on the path with probability Δ^{-1}. Therefore, the interference length is implicitly assumed to be $\frac{\Delta}{\Delta-1}$, which is clearly an inappropriate assumption for a ring network.

We now propose a general traffic model appropriate for a wider variety of networks, including rings. Afterwards, we will calculate the blocking probability along an H hop path with and without wavelength changers. Then we will be in a position to continue our discussion on the benefits of wavelength changers.

[6]For instance, in [3], the authors simulated a wavelength changing gain (measured in a related but slightly different way than ours) for 1024 and 16 node random networks with $\Delta = 4$, $F = 10$, and blocking probability .01. They predicted a utilization gain of 1.4 for the 1024 network and measured no gain in efficiency for the 16 node network. Using $H \approx log_\Delta N$, our models predict a maximum gain of 2.2 for $N = 1024$ and 1.1 for $N = 16$.

Consider Fig. 1 again. We model the traffic along this path at the time of a session request from A to B. As before, we assume that there are no sessions on the access and/or exit fibers. At any node $i \geq 1$, sessions can leave or join the path. A session using hop i which also uses hop $i-1$ is called a *continuing* session of hop i. A session using hop i which has joined the path at node i is called a *new* session of hop i. The model is based on a pair of parameters: $P_l > 0$, the probability a session leaves the path at a node, and $P_n > 0$, the probability a new session joins the path at a node on an *available* wavelength. A wavelength is called available on hop i if either it is unused on hop $i-1$ or the session using it on hop $i-1$ leaves. The model is formally defined by six assumptions. Note that below a session refers to an interfering session and $1 \leq i \leq H$.

1. *All events on different wavelengths are statistically independent.*

Therefore, it is sufficient to describe the rest of the model for an arbitrary wavelength λ.

2. *The probability that λ is used on hop 0 is $\rho_0 \stackrel{\text{def}}{=} 0$.*
3. *Given the state of hop $i-1$, i.e. if λ is used or not, the state of hop i is statistically independent of the states of hops $0, 1, 2, .., i-2$.*
4. *Given that λ is used on hop $i-1$, we assume that the session on λ leaves the path at node i with probability P_l. Else, it continues on the same wavelength. All sessions leave the path at node $H+1$.*
5. *If λ is not used on hop $i-1$, then a new session joins at node i on λ with probability P_n. No sessions join at node $H+1$.*
6. *If λ is used on hop $i-1$ and the session using λ on hop $i-1$ leaves at node i, then a new session joins at node i on λ with probability P_n. No calls join at node $H+1$.*

Notice that we have assumed that interfering continuing sessions remain on the same wavelength. This is of course the case in networks without wavelength changers, but it might seem at first thought an inappropriate assumption otherwise. Clearly, in networks with wavelength changers, it is not necessary for a session to use the same wavelength on hop i as on hop $i-1$. However, we argue that there is no loss in generality in assuming that the wavelengths of continuing sessions are not changed. To see this, suppose m sessions continue from hop $i-1$ to hop i and that n new calls enter at node i, $m+n \leq F$. Then with wavelength changers, we can without any loss in performance only change the wavelengths of the new sessions and leave the wavelengths of the continuing sessions unchanged.[7] Note that this is not the same as assuming that a session uses the same wavelength on each hop of the path it occupies. For example, consider the case where a session on λ leaves the path at node i only to rejoin it at node $k > i$. Then the wavelength of this session may have to be changed to avoid a conflict with a session continuing from hop $k-1$.

[7] This is possible to do for any path, but certainly not for all paths simultaneously.

The above discussion applies only to the interfering sessions at the time A and B make a request. Therefore, the blocking probabilities are still defined the same. In particular, in networks with wavelength changers the request is blocked only if a fiber is full. In networks without changers, the call is blocked if no wavelength is free on all the hops.

Notice that

$$Pr\,(\lambda \text{ used on } i | \lambda \text{ not used on } i-1\ ) = P_n \quad (20)$$

$$Pr\,(\lambda \text{ used on } i | \lambda \text{ used on } i-1) = (1-P_l) + P_l P_n \quad (21)$$

where the first equation is the definition of P_n and the second equation follows from the fact that if λ is used on hop $i-1$ then it is used on hop i if the call continues on the path or if it leaves and a new call arrives. Notice that $P_l = 1$ implies that a wavelength is used on successive links independently. This corresponds to the model of Section 2. Also, if $P_n = (1-\frac{1}{\Delta})q$, and $P_l = 1 - \frac{1}{\Delta}$, then $1 - P_l + P_l P_n = \frac{1}{\Delta} + q\left(1 - \frac{1}{\Delta}\right)$, and we get the model of Section 3. We will further discuss these relationships later.

From assumptions 1. and 2., the number of calls on hop i is binomial distributed. The probability ρ_i that a wavelength λ is used on hop i is easily calculated; if λ is used on hop $i-1$, it is used on hop i with probability $1 - P_l + P_l P_n$, otherwise it is used with probability P_n. Therefore, $\rho_i = (1 - P_l + P_l P_n)\rho_{i-1} + P_n(1-\rho_{i-1})$ and

$$\rho_i = \rho \left\{1 - [1 - (P_l + P_n - P_l P_n)]^i\right\} \quad (22)$$

where

$$\rho \stackrel{\text{def}}{=} \frac{P_n}{P_n + P_l - P_n P_l} \quad (23)$$

Note that $P_l + P_n - P_l P_n$ is the probability that at a node, a new call joins or an old call leaves. If a new call joins or an old call leaves then the node is either adding or removing a call, i.e. there is mixing. We will see that the benefit of wavelength changers increases with this probability since increasing P_l or P_n increases the average number of interfering calls (increasing P_l increases the expected number of available wavelengths at a node, and therefore increases the expected number of new calls).

Since $P_l + P_n - P_l P_n > 0$, ρ_i increases with i towards ρ. Furthermore, if this probability is large, the convergence is rapid and the average utilization along the path is $\approx \rho$. For these reasons, we continue to call ρ the utilization.

Before using eqns. (20), (21), and (22) to derive the blocking probability with and without changers, we discuss how P_n and P_l might be determined when the interference length L and the traffic between users is known. Suppose a session joins the path on λ at node i. Then with probability P_l, the session leaves the path at node $i+1$. The expected number of hops used by this session is

$$\sum_{h=i}^{H}(h-i+1)(1-P_l)^{h-i}P_l \approx \frac{1}{P_l} \qquad (24)$$

where the approximation is valid as long as $i+L << H$. That is, the approximation ignores the truncation effects caused by the finite length of the path. We will assume this approximation is valid for the rest of this paper and set $P_l = 1/L$. Now from eqn.(23),

$$P_n = \frac{\rho P_l}{1-\rho(1-P_l)} = \frac{\rho}{L-\rho(L-1)} \qquad (25)$$

so P_n can be determined from L and ρ. A reasonable estimate for ρ is

$$\rho = \frac{N\gamma\bar{H}}{Ff} \qquad (26)$$

where N is the number of users, γ is the expected number of active outgoing calls per user, $\bar{H}$ is the expected number of hops/call, F is the number of wavelengths, and f is the total number of fibers in the network.

To illustrate our approach, consider an N node *bidirectional* ring network with a load of γ Erlangs per node. The average interference length for a call request traveling half-way around the ring is about $N/4$; therefore, we set $P_l = 4/N$. The utilization is $\rho = \gamma N/8F$. Therefore, $P_n \approx \frac{2\gamma}{F-.125\gamma N}$. Note that $.125\gamma N$ is the expected load per link, so that even in networks with wavelength changers, we would expect $F >> .125\gamma N$. We will use these numbers below to estimate the performance of a ring network and the gain in utilization that wavelength changers can provide.

It is a simple matter to calculate the blocking probability without wavelength changers, i.e. $P_b =$

$$\left[1-\prod_{i=1}^{H} Pr\{\lambda \text{ free on } i \mid \lambda \text{ free on } 0,1,...i-1\}\right]^F$$

which reduces to

$$\begin{aligned} P_b &= \left[1-\prod_{i=1}^{H} Pr\{\lambda \text{ free on } i \mid \lambda \text{ free on } i-1\}\right]^F \\ &= \left[1-(1-P_n)^H\right]^F \qquad (27) \end{aligned}$$

As can be seen, the path blocking probability in a network without wavelength changers is directly dependent on the probability a new call joins the path P_n, and only indirectly dependent on the utilization ρ (through eqn. (23)). Therefore, if the interference length is large, it is possible to have a very large utilization $\rho \approx 1$ and still have a very small blocking probability. To see this more clearly, invert eqn. (27) for P_n and use eqn. (23). Then the achievable utilization p for a given blocking probability is

$$p = \frac{P_n}{P_n+P_l-P_nP_l} \qquad (28)$$

where

$$P_n = 1-\left(1-P_b^{1/F}\right)^{1/H} \qquad (29)$$

We see that for small P_l, i.e. when calls tend to stay together, the utilization can approach 1.

With wavelength changers, the blocking probability is

$$\begin{aligned} P_b' &= 1-\prod_{i=1}^{H} Pr\{\bar{f}_i \mid \bar{f}_{i-1},\bar{f}_{i-2},...\bar{f}_0\} \\ &= 1-\prod_{i=1}^{H} Pr\{\bar{f}_i \mid \bar{f}_{i-1}\} \\ &= 1-\prod_{i=1}^{H}\left[1-\frac{Pr(f_i)-Pr(f_i|f_{i-1})\,Pr(f_{i-1})}{1-Pr(f_{i-1})}\right] \\ &= 1-\prod_{i=1}^{H}\left[1-\frac{\rho_i^F-(1-P_l+P_lP_n)^F\rho_{i-1}^F}{1-\rho_{i-1}^F}\right] \qquad (30) \end{aligned}$$

where we have used

$$\begin{aligned} Pr(f_i) &= Pr(f_i \mid f_{i-1})\,Pr(f_{i-1}) \\ &\quad + Pr(f_i \mid \bar{f}_{i-1})\,Pr(\bar{f}_{i-1}) \\ Pr(f_i \mid \bar{f}_{i-1}) &= 1-Pr(\bar{f}_i \mid \bar{f}_{i-1}) \\ Pr(f_i \mid f_{i-1}) &= [Pr\{\lambda \text{ free on } i \mid \lambda \text{ free on } i-1\}]^F \\ &= (1-P_l+P_lP_n)^F \end{aligned}$$

The third equality follows from the assumed independence of wavelengths and assumption 4.

Bounds on P_b' can be obtained as follows. P_b' is at least the probability that hop H is full, i.e. $P_b' \geq \rho_H^F$. Also, P_b' is the probability that some link is full, which is no more than $\sum_{i=1}^{H}\rho_i^F$. The sum is no more than $H\rho_H^F \leq H\rho^F$ since the ρ_i's are increasing to ρ. Therefore,

$$\rho\left\{1-[1-(P_n+P_l-P_nP_l)]^H\right\} \leq (P_b')^{1/F} \leq H^{1/F}\rho$$

Now a good approximation for ρ can be obtained if $H^{1/F} \approx 1$ and if $H > 2L$. First, since $1-(P_n+P_l-P_nP_l) \leq 1-P_l = 1-\frac{1}{L}$,

$$\rho\left[1-\left(1-\frac{1}{L}\right)^H\right] \leq (P_b')^{1/F} \leq H^{1/F}\rho \qquad (31)$$

Now if $H > 2L$, $(1-\frac{1}{L})^H \leq e^{-2} = .13$. So for most cases of interest, the achievable utilization in a network with wavelength changers q is approximately $(P_b'/H)^{1/F}$, which is also approximately

$$q \approx \left[1-(1-P_b')^{1/H}\right]^{1/F} \qquad (32)$$

Even if $L = H/2$, the error is only about 13 % for large F.

Eqns. (27) and (30) can be used to estimate blocking probabilities when P_l and P_n are known. Our goal here is to estimate the performance gain of wavelength changers for the same path and routing algorithm (same L). We use $G = \frac{q}{p}$, where, as before, q and p are the achievable utilizations for networks with and without wavelength changers for the same blocking probability, same number of wavelengths, and same interference length L.[8] From eqns (29) and (32) with $P_b' = P_b$, we get after simplification

$$G \approx \frac{\left[1-(1-P_b)^{1/H}\right]^{1/F}}{1-\left(1-P_b^{1/F}\right)^{1/H}} \times \left\{\left[1-\left(1-P_b^{1/F}\right)^{1/H}\right]\left(1-\frac{1}{L}\right)+\frac{1}{L}\right\} \qquad (33)$$

Eqn. (33) compares a system with wavelength changers to a system without wavelength changers for the same blocking probability and same interference length. Since the two systems have different utilizations, the new call probability P_n will be different in the two systems. Let $P_n = 1-\left(1-P_b^{1/F}\right)^{1/H}$ be the new call probability for a system without wavelength changers. Then G can be expressed as

$$\begin{aligned} G &\approx \frac{\left[1-(1-P_b)^{1/H}\right]^{1/F}}{1-\left(1-P_b^{1/F}\right)^{1/H}}\,(P_n+P_l-P_nP_l) \\ &= G_o\,(P_n+P_l-P_nP_l) \qquad (34)\end{aligned}$$

where G_o is the gain when the interference length $L = 1$, or equivalently the gain when successive links are assumed independent. The gain increases with Pr (new call joins or old call leaves) since increasing this probability increases mixing. A plot of G is shown in Fig. 8 for a 20 hop path, a blocking probability of 10^{-3}, and interference lengths of $L = 1, 2, 4$. As can be seen, the gain is proportional to $1/L$. Notice the similarity between these curves and the curves for $H = 20, 10, 5$ shown in Fig. 4a. In terms of the gain, an H hop path with interference length L looks like an H/L hop path.

If P_n is small (low P_b without changers), the gain is reduced by about $1/L$. For example, we earlier calculated $P_l = 4/N$ and $P_n \approx 2\gamma/(F - .125\gamma N)$ for a request traveling half-way around an N node ring network with load γ Erlangs per node. For these values, $G \approx 4G_o/N$ for $F >> .125\gamma N$. Since $G_o \lesssim H/2 = N/4$ in this case, our model predicts no statistical advantage for wavelength changers in a ring network. Of course it would not be surprising if changers provided a small gain in efficiency as many of our calculations have been rough and the model only approximates reality.

[8] This definition is consistent with the previous two sections where the interference length was 1 and $\Delta/(\Delta-1)$.

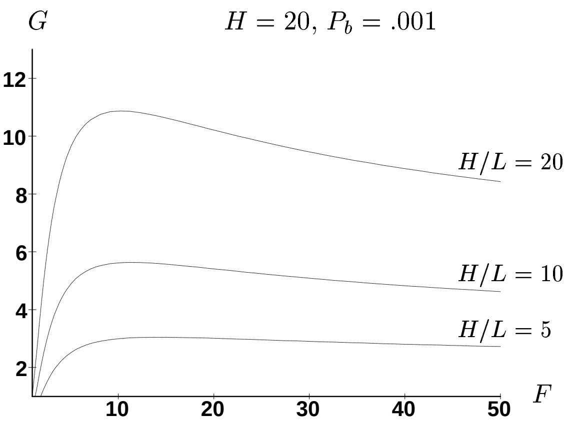

Figure 8: **Effect of Interference Length on the gain.**

For large P_n, the gain is not reduced much below G_0. However, in order to achieve large P_n for small P_b, the number of wavelengths must be large, and therefore G_0 is not near its maximum.

Intuitively, wavelength changers help only if bandwidth is underutilized on a link (since the gain is the increase in utilization) and if calls have lots of interferers (else many wavelengths would be unused and the utilization could be increased). However if calls have lots of interferers and if interfering calls tend to stay together, it is impossible to greatly underutilize a link.

The following theorem is another indication of the importance of interference length.

Theorem 1 *Consider a network with diameter D without wavelength changers routing a set of calls. Suppose also that each call which interferes with another call does so on exactly L links and that F is the minimum number of wavelengths required to avoid collisions. Then for the same set of calls over the same routes, a network with wavelength changers requires at least FL/D wavelengths.*

Proof. *Consider an arbitrary call s. Let P_s be the path of s and let $I_s - 1$ be the number of interfering calls of s on this path. Then at least one s has $I_s \geq F$, else $F-1$ wavelengths would suffice. For this s, notice that the I_s calls use a bandwidth of $(I_s-1)L + H$ on the H hops of P_s, i.e. the interfering calls use 1 wavelength on L hops and s uses 1 wavelength on H hops.*

Therefore, at least one fiber carries $(I_s-1)\frac{L}{H}+1$ calls and a network with wavelength changers *requires at least this many wavelengths. Since $I_s = F$ and since $L \leq H$, at least FL/H wavelengths are required. Since $H \leq D$, the statement followings.* $\square$

The theorem can be extended in many ways, e.g. to make statements about average interference length, network wide efficiency improvements, etc. These results will be published separately.

The main limitation is that we using the same routes in networks with and without wavelength changers. It may be that for a topology, a good routing in the wavelength changing case has a small interference length and a large interference length for the non-changing case. In this case, the theorem would provide little insight into the benefits of changers for that topology.

5 Conclusions

We modeled the probability of the path being blocked with and without wavelength changers under simple traffic models.

The blocking probability with and without wavelength changers increase with the number of hops H. However, the effect is much more dramatic in networks without wavelength changers since the number of calls a given call shares *some* link with tends to increase with H. That is, networks with large diameter D tend to have a lot of mixing. It therefore becomes harder to find a wavelength which is not used by any interfering call. This has lead researchers to conclude that minimizing the network diameter and employing minimum hop routing are reasonable heuristics for networks without wavelength changers. We concur with two caveats.

First, in regular networks such as the DeBruijn Networks, the amount of call mixing tends to increase with the switch size Δ. That is, the number of interfering calls tends to increase with Δ as well as H. We therefore expect P_b to increase with Δ and H. Since the diameter of a network tends to decrease with increasing switch size, there is a topological design trade-off between Δ and D.

Second, networks with large interference length L have smaller blocking probability than networks with small interference length. In particular, the *effective path length* H/L is the most important parameter. We estimated a gain in fiber utilization using wavelength changers of no more than about $H/2L$. Therefore, we might choose to design networks with larger diameter if this permitted us to increase the interference length. Also, for a given topology, we may choose not to do minimum hop routing if this allowed us to decrease H/L.

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References

[1] R. Barry, *Wavelength Routing for All-Optical Networks.* PhD thesis, Department of Electrical Engineering and Computer Science, MIT, 1993.

[2] A. Aggarwal, A. Bar-Noy, D. Coppersmith, R. Ramaswami, B. Schieber, and M. Sudan, "Efficient routing and scheduling algorithms for optical networks," tech. rep., IBM Research Report RC 18967 (82799), June 1993.

[3] R. Ramaswami and K. N. Sivarajan, "Routing and wavelength assignment in all-optical networks," tech. rep., IBM TJ WATSON RESEARCH CENTER, 1993.

[4] C. Lee, "Analysis of switching networks," *Bell System Technical Journal*, vol. 34, pp. 1287–1315, Nov. 1955.

[5] K. Sivarajan and R. Ramaswami, "Lightwave networks based on debruijn graphs," *IEEE Transactions on Networking*, vol. 2, February 1994.

[6] N. Pippenger, "On crossbar switching networks," *IEEE Transactions on Communications*, vol. 23, pp. 646–659, June 1975.

[7] J. Y. Hui, *Switching and Traffic Theory for Integrated Broadband Networks.* Massachusetts: Kluwer Academic Publishers, 1990.